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Plane
Geometry

Plane
Geometry
Suggestive Method

By
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Whitewater, Wisconsin

Atkinson, Mentzer & Grover
Chicago Publishers Boston

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THE PREFACE.

THIS present book is a revision of the Van Velzer and Shutts Plane and Solid Geometry, Suggestive Method, published by Tracy, Gibbs & Company. The method of the book was used by the author for several years from mimeograph reprints. This text was then revised and incorporated into text-book form. After being again thoroughly tested in many of the best schools in the country, the work has been again revised in an attempt to make it more suggestive to the teacher and more helpful to the student.

In putting the work into its present form the scientific classification of the subject-matter has been departed from when it was thought that by so doing the work could be better graded to the ability of the average pupil. For this reason the subject of the triangle has been introduced before the relation of lines and angles has been fully discussed.

The treatment of the theory of measurement has been modified to make it more easily understood. A treatment of the application of proportion has been suggested that will make the pupil more independent in his work, and at the same time it has not increased the difficulty of the subject.

The book, as now arranged, is sufficient for all college entrance requirements, yet it can be completed, including all the exercises, by high-school pupils in one school year. For those schools that devote a year and a half to the subject additional work has been placed in the Appendix, to which references are given in the text, so that the various propositions and exercises can be taken up in logical order. A fuller treatment of the theory of limits, which many teachers desire, is also given in the Appendix.

A departure from ordinary methods will be noticed in the treatment of proportion. It has not been thought wise to follow the usual method of limiting the subject to proportions whose terms are pure numbers, nor yet to follow the Euclidian method common in England, which admits of proportions whose terms are concrete magnitudes, but which is so difficult that it can be understood by only the best students The method in the text will be found to admit of proportions whose terms are concrete magnitudes, yet it will do no violence to the fundamental ideas of Arithmetic regarding operations upon concrete magnitudes. It is believed that the subject of limits is treated in so simple a manner that beginners can grasp it.

An edition, consisting of the theorems, the diagrams, the things given, and the things to prove, *without* suggestions for demonstration, will accompany the book for class-room use. It is hoped that this will be found as valuable in Geometry teaching as "text editions" have been in teaching the classics. This "Class-Room Edition" will save the time of the recitation usually consumed by the pupils in drawing the figures upon the blackboard. The complete edition of the book can be banished from the recitation and the temptation to get assistance from the text eliminated. The blackboards can thus be reserved for original demonstrations of exercises and for suggestive work by the teacher. The Class-Room Edition will lengthen the daily recitation, and make it possible for more work to be done in a year.

The author wishes to thank those who have given the previous edition of the book so kind a reception. It is hoped that the present book will more fully meet the needs of all teachers who wish their pupils to make the largest possible growth in independent thinking in Geometry.

Acknowledgment should be made of the scholarly criticism of Dr. C. A. Van Velzer, *Head of the Department of Mathematics in the University of Wisconsin,* for his helpful services in preparing for publication the manuscript of the first edition of the book.

Thanks are also due for valuable suggestions in reading the proof to Mr. G. E. Bunsa, *Superintendent of Schools, Columbus, Wisconsin;* Mr. R. L. Sandwick, *Principal of the Deerfield Township High School, Highland Park, Illinois;* Miss Genevieve Decker, *Teacher of Mathematics in the High School, Janesville, Wisconsin;* Miss Maud Averill, *Teacher of Mathematics in the High School, Whitewater, Wisconsin,* and to Mr. Frank P. Dodge, *Instructor in Mathematics in the Roxbury Latin School, Roxbury, Massachusetts.*

G. C. S.

Whitewater Wis., August 25, 1904.

SUGGESTIONS TO TEACHERS.

GEOMETRY is essentially a disciplinary study. The value derived from its study is in proportion to the amount of independent thought expended by the pupil. A text-book in Geometry is in the nature of a "key" to the extent to which the demonstrations are written out for the pupil. That part of the work which a pupil can do for himself should not be done for him. The teacher and text-book should furnish the pupil with data and stimulate thought rather than give him a set form of words which he may repeat verbatim, with or without, the ideas which these words should express.

In this Geometry suggestions arranged in logical order take the place of detailed demonstration. These suggestions are intended to stimulate and direct the thought of the pupil so that he may largely work out his own demonstrations.

Model demonstrations are given of a few propositions to show the student the *form* in which they should be presented. The answers to the suggestions, logically arranged, constitute the demonstration. The suggestions should be studied in the order given, for each suggestion usually depends upon the preceding one. The answer to a suggestion should consist of a statement of the relations asked for, together with the authority in full for such statement.

To permit the pupil to ignore the authority is to encourage carelessness, slovenliness, and inaccuracy in demonstration. A common error is to apply authority that does not exactly fit the conditions under consideration. The pupil must understand that the authority should, without exception, be a definition, an axiom, or a previously proved proposition. "It seems so," or, "it looks reasonable," or any expression of judgment will not do. The pupil should be encouraged to search out his own authority, even when the authority is quoted for him in the suggestions, and to use the reference simply for verification. A pride in independent work is a most important factor in securing satisfactory results.

In the preparation of the lesson the pupil should write out his demonstration, noting carefully the form of the "models." This will insure correct form and avoid haziness of thought. During the first few weeks this written work, as well as tests taken in the recitation, should be read by the teacher and returned to the pupil for correction.

The exercises, or at least a part of them, should be demonstrated daily along with the propositions as they occur, and not be studied all together at the end of a chapter.

The best results will be obtained by starting slowly, reviewing frequently, and passing over nothing that is not clearly understood. Since each demonstration involves previous propositions and definitions, facility in demonstration can best be secured by committing to memory, each theorem, definition and axiom; for that authority cannot be readily recognized and applied which is imperfectly remembered. The demonstrations should not be committed to memory.

The subject of proportion is probably the most difficult part of Geometry. Clearness of thought in the applications of proportion can be obtained only by careful illustration and rigid demonstration in the theory. To teach the theory of proportion by means of numbers, and then to apply the principles developed to geometric magnitudes and numbers indiscriminately without consideration of limitations of the various statements, is not scientific. Note 262, page 137, should receive careful attention. In deriving the form $A = m\ B$ from $\frac{A}{B} = m$ the tendency is to claim the multiplication of both members of the equation by B. This is correct if B is a number, but the process is unthinkable if B is a geometric magnitude. $\frac{A}{B}$ means that A is divided or measured by the unit B, hence to say that A contains B, m times, is simply another way of saying that A is m times the unit B, or m B. 12 contains 4 three times ($\frac{12}{4} = 3$) is another form of expression for 12 is equal to 3 fours ($12 = 3 \times 4$). The expression $\frac{1\text{ foot}}{1\text{ inch}} = 12$, means the same as the expression 1 foot is equal to 12 inches. In this connection see § 270.

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PLANE GEOMETRY.

CHAPTER I.

RECTILINEAR FIGURES.

Definitions.

1. The block represented in the accompanying figure occupies a limited portion of space. If we imagine the block to be removed, its form or shape can still be retained in the mind. This is true of any object or body.

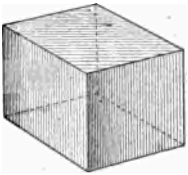


FIG. 1.

The space conceived to be occupied by an object or body as distinguished from the substance of which it is made, is a **geometrical solid.** The matter or substance of which a body or object is composed is a physical solid. Hence a geometrical solid is the shape or form of a physical solid, or some form or figure conceived by the mind.

A **geometrical solid** is a limited portion of space, and has length, breadth, and thickness.

The term solid will be used hereafter to signify a geometrical solid.

2. When space is divided into distinct portions or geometrical solids, the boundaries of these portions or solids are **surfaces.** Distinct portions of the bounding surface are **faces.**

Surface has length and breadth, but no thickness.

3. When a surface is divided into distinct portions, the boundaries of these portions are **lines.** In the solid, represented in Fig. 1, the edges, or boundaries of the faces, are lines. These lines, being the intersection of faces which have no thickness, can themselves have neither breadth nor thickness.

A **line** has length, but neither breadth nor thickness.

4. When a line is divided into distinct portions, the limits of these portions are **points.** In the solid, represented in Fig. 1, the corners, or limits of the edges, are points. These points, being the intersections of lines which have neither breadth nor thickness, can themselves have neither length, breadth, nor thickness.

A **point** has position, but neither length, breadth, nor thickness.

5. A surface can be conceived of apart from a solid, a line apart from a surface, and a point apart from a line. If a point is conceived to move, the path in which it moves is a line. Hence a line is the path, or **locus,** of a moving point.

A line can be thought of as generated by a **point in motion;** surface can be thought of as generated by a **line in motion;** a solid, as generated by a **surface in motion.**

6. A **geometrical figure** is a combination of points, lines, surfaces, or solids.

Geometrical figures are ideal, that is, they are mental conceptions, but they can be represented to the eye only by material substances. For instance, a line can be represented by a mark made by a pencil or crayon; a solid can be represented by a drawing, by a block of wood, or by some other material of any given shape.

To avoid multiplying words, the material representation of geometrical figures will be generally referred to as standing for the mental conceptions themselves. The pupil will be able to tell by the context whether the word "figure" refers to a geometrical figure or to the material representation of a figure.

7. A **straight line** is a line such that any part of it, however placed, lies wholly in any other part if its extremities lie in that part. Let O B, which is any part of A B, be placed upon some other part in any way, except that O and B shall lie upon that part, for instance with O at M and B at N. If O B exactly coincides with M N, A B is a straight line. Illustrate by lines represented by wood, paper, or other material.

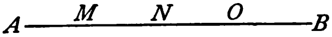


FIG. 2.

A line is **read** by naming letters placed at its extremities, as line A B in Fig. 3; or by naming a single letter placed upon it, as line O in Fig. 9.

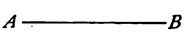


FIG. 3.

8. A **broken line** is a line made up of a succession of different straight lines, as A B C D E, in Fig. 4.

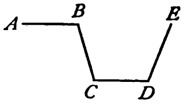


FIG. 4

9. A **curved line,** or a **curve,** is a line no portion of which is straight, as C D, in Fig. 5.

FIG. 5.

10. A **plane surface,** or a **plane,** is a surface such that if any two of its points be joined by a straight line the line lies wholly in the plane surface. If a carpenter wishes to determine whether or not the surface of a board

is a plane, he tries to place a straight edge so that at least two of its points touch the surface. If his straight edge lies continuously in the surface, it is a plane surface.

11. A **plane figure** is a figure that lies wholly in the same plane.

12. A plane figure, in which the lines are all straight lines, is a **rectilinear figure.**

13. **Magnitudes** are figures considered only with reference to extent.

14. **Geometry** is the science that treats of points, lines, surfaces, and solids, and is concerned with the construction and measurement of geometrical figures.

15. **Plane geometry** treats of plane figures.

16. **Solid geometry** treats of figures which are not wholly in the same plane.

Angles.

17. When two straight lines meet or intersect, they contain, or make with each other, **an angle.**

The two lines are the **sides** or **arms** of the angle, and the point of meeting is its **vertex.**

(*a*) An angle can be **read** by naming the letter at the vertex of the angle between the letters upon the sides of the angle, as angle $A\ B\ C$ or angle $C\ B\ A$ in Fig. 6. When there is only one angle at a given vertex, it is sufficient to read the letter at the vertex, as angle B. When two or more angles have a common vertex, letters or figures are frequently placed near the vertex between the sides of the angles to designate the angles, as *m* and *n* in Fig 7. For example,

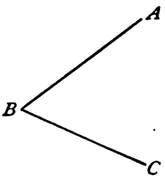


FIG. 6.

we say angle *m* instead of angle *C B A*, and angle *n* instead of angle *C B D*.

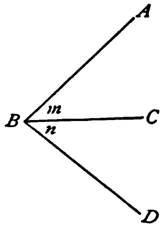


FIG. 7.

(*b*) When a line, coincident with one side of an angle, revolves about the vertex remaining always in the same plane until it arrives at the position of the other side of the angle, the line **turns through the angle,** and the greater the amount of turning the greater the angle.

(*c*) Hence the **magnitude** of an angle depends upon the amount of revolution necessary to turn a line through the angle. The length of the sides of the angle bears no relation to the size of the angle. The magnitude of an angle may be made clear by means of a pair of dividers, the legs of the dividers representing the sides of the angle and the hinge, the vertex. If the dividers are opened a given amount, a certain angle is represented; if from that position they are closed more or less, a smaller angle is represented; if they are opened farther, a larger angle is represented.

(*d*) A line can turn through an angle in two directions, hence there are two angles which have the same sides and the same vertex.

For example, the side *O B*, in Fig. 8, can be made to turn through the angle *m* by a motion opposite to that of the hands of a watch, to the position *O A*, or it can be made to turn through the angle *n* by a motion like that of the hands of a watch, to the same position *O A*.

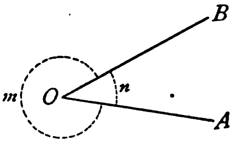


FIG. 8.

Two angles which have the same

sides and the same vertex are **conjugate angles.** When an angle is referred to, the smaller of the two conjugate angles is always meant, unless the other is specifically mentioned.

(*e*) The **direction of a line** is its position as determined by the angle it makes with a given line upon a given side of it. For instance, the direction northeast means a line which makes an angle of forty-five degrees with a south to north line on the east side of it.

A surveyor indicates a particular direction when he says "south 17° 20′ east." He means the line which makes an angle of 17° 20′ with the north and south line on the east side of it.

18. When the two sides of two conjugate angles lie in the same straight line, each conjugate angle is a **straight angle;** for example, the angle $A\ O\ B$ in Fig. 9 is a straight angle.

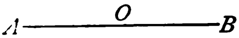


FIG. 9.

The two sides of a straight angle form a straight line.

19. Two angles which have a common vertex and one common side, and are on opposite sides of this common side, are **adjacent angles.** In Fig. 10, angles *1* and *2* are adjacent angles.

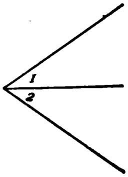


FIG. 10.

20. A **right angle** is an angle made by two straight lines which meet so that the adjacent angles formed are equal. In Fig. 10, if $A\ O$ and $B\ C$ meet so that the adjacent angles *1* and *2* are equal, angle *1* is a right angle; angle *2* is also a right angle.

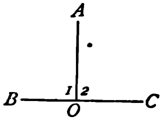


FIG. 11.

21. An **acute angle** is an angle that is less than a right angle. In Fig. 12, *B O C* is an acute angle.

22. An **obtuse angle** is an angle that is greater than a right angle. In the same figure, *A O B* is an obtuse angle.

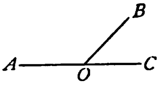


FIG. 12.

Acute and obtuse angles are **oblique angles.**

23. A **perpendicular line,** or a **perpendicular,** is one that makes right angles with another line. In Fig. 11, *A O* is perpendicular to *B C*, and *B C* is perpendicular to *A O*.

24. A line is **oblique** to another line when it makes oblique angles with that line. The two lines are sometimes called oblique lines. In the Fig. 12, *B O* is oblique to *A C*, and *A C* is oblique to *B O*.

25. When two lines intersect, the opposite angles are **vertical angles.** The angles *m* and *n* are vertical angles; angles *p* and *o* are also vertical angles.

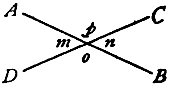


FIG. 13.

26. Two angles are **complementary** if their sum equals one right angle. The angles are then **complements** of each other.

If in Fig. 14 *M B* is perpendicular to *A D*, angles *1* and *2* are complements of each other. If angles *A* and *B*, Fig. 15, are together equal to one right angle, they are complements of each other.

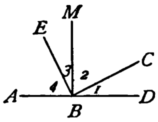


FIG. 14.

27. Two adjacent angles that are complements of each other are **complementary adjacent angles,** as angles *1* and *2*, Fig. 14.

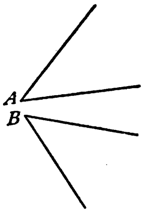


FIG. 15.

28. Two angles are **supplementary** if their sum equals two right angles. The angles are then **supplements** of

each other. Angle *1* and angle *A B C*, in Fig 14, and the angles *m* and *p*, in Fig. 13, are supplementary angles. If in Fig. 15 the sum of angles *A* and *B* is equal to two right angles, *A* and *B* are supplementary angles.

29. Two adjacent angles which are supplements of each other are **supplementary adjacent angles.**

Logical Terms.

30. A **theorem** is a truth which requires demonstration. For example: *If two straight lines intersect each other, the vertical angles are equal.*

31. The statement of a theorem is its **enunciation,** or the **general enunciation.**

When a drawing is made to illustrate a theorem, the description of the drawing is the **special enunciation.**

32. A theorem consists of two parts, the **hypothesis** and the **conclusion.** The conditional part of a theorem is the **hypothesis.** For example, in the above theorem the hypothesis is: "*If two straight lines intersect each other.*" The hypothesis is sometimes called the **premises.**

The truth depending upon, or following from, the hypothesis is the **conclusion.**

In the theorem stated in article 30, the truth, "*the vertical angles are equal,*" depends upon the hypothesis, "*If two straight lines intersect each other,*" and is therefore the conclusion.

33. The **demonstration,** or **proof,** of a theorem is the course of reasoning by which the truth of the theorem is established.

34. A **problem** is a question proposed for solution, or the statement of certain relations which are to be pro-

duced. For instance: *To construct a line perpendicular to a given line at a given point.*

35. A **proposition** is a general term for a theorem or a problem.

36. A **corollary** is a proposition easily deduced from the proposition to which it is attached, with the aid, if necessary, of one or more previous propositions.

37. A **scholium** is a remark upon one or more propositions with respect to their applications, limitations, or connections.

38. An **axiom** is a truth which, from its simplicity, must be admitted without demonstration: as, *The whole of anything is equal to the sum of all its parts.*

39. A **postulate** is a proposition which, for the sake of argument, is admitted without demonstration or solution: as, §214, or *Let it be granted that a straight line can be drawn between two points.*

Axioms.

40. *1.* Things which are equal to the same thing or equal things, are equal to each other.

2. If equals are added to equals the sums are equal.

3. If equals are subtracted from equals the differences are equal.

4. If equals are multiplied by the same number or by equals the products are equal.

COROLLARY.—Doubles of equals are equal.

5. If equals are divided by the same number or by equals the quotients are equal.

COROLLARY.—Halves of equals are equal.

6. If equals are added to unequals the sums are unequal, and that sum is the greater which is obtained by adding to the greater magnitude.

COROLLARY.—If unequals are multiplied by equals the products are unequal, and that product is the greater which is obtained by multiplying the greater magnitude.

7. If equals are subtracted from unequals the differences are unequal, and that difference is the greater which is obtained by subtracting from the greater magnitude.

COROLLARY.—If unequals are divided by equals, the quotients are unequal, and that quotient is the greater which is obtained by dividing the greater magnitude.

8. If unequals are subtracted from equals the differences are unequal, and that difference is less which is obtained by subtracting the greater magnitude.

9. The whole is greater than any of its parts.

10. The whole is equal to the sum of all its parts.

11. A straight line is the shortest distance between two points.

12. If two straight lines have two points in common they are one and the same straight line.

COROLLARY 1.—Two straight lines can intersect in but one point.

COROLLARY 2.—But one straight line can be drawn between two points.

13. Magnitudes which coincide are equal in all respects, and conversely, magnitudes which are equal may be made to coincide.

14. Every magnitude has two halves or is equal to two halves of itself.

COROLLARY.—Every magnitude, however small, may be divided into two or more parts.

15. If any magnitude, however small, is added to one of two equal magnitudes, and the same subtracted from the other, the results are unequal.

16. If A, B and C are three magnitudes, and if A is less than B and B is less than C, A is less than C.

17. A magnitude which is less or greater than one of two equal magnitudes is less or greater, respectively, than the other.

18. Only one line can be drawn through a given point parallel to a given line.

Postulate 1.—A line can be revolved about a given point until it embraces another point or takes the direction of a line drawn through the given point.

Postulate 2.—A figure can be thought of as being changed in position without making any change in the relation of its parts.

Symbols and Abbreviations.

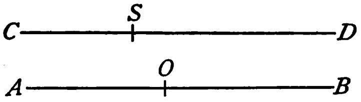
41. ∠ —*angle*
∠s—*angles*
⊥ —*perpendicular*
⊥s—*perpendiculars*
‖ —*parallel*
‖ s—*parallels*
△ —*triangle*
△s—*triangles*
▱ —*parallelogram*
▱s—*parallelograms*
⊙ —*circle*
⊙s—*circles*
∴ —*therefore.*

Ax.—*axiom.*
Cons.—*construction.*
Cor.—*corollary.*
Def.—*definition.*
Ex.—*exercise.*
Auth.—*authority.*
Hyp.—*hypothesis.*
Rt.—*right.*
Sch.—*scholium.*
St.—*straight.*
Sug.—*suggestion.*
Sugs.—*suggestions.*
P.—*Postulate.*

Q. E. D.—*Which was to be demonstrated.*

PROPOSITION I.*

42. Theorem. *Two straight angles are equal.*



Let A O B and C S D represent two straight angles.

To prove that angle A O B is equal to angle C S D.

Suggestion 1. What kind of a line do the two sides of $\angle A O B$ form? Of $\angle C S D$? § 18.

2. Place $\angle A O B$ so that point O lies upon S, and another point of line $A O B$ upon line $C S D$. § 40, P. 2.

3. Where does $\angle A O B$ lie? *Ax.* 12.

4. How then does the amount of revolution in turning through $\angle A O B$ (§ 17 *c*), compare with that in turning through $\angle C S D$? *Ax.* 13.

Therefore—

What is the hypothesis in this theorem?

What is the conclusion?

What is the general enunciation? The special?

Ex. 1. Given three points not in a straight line, how many straight lines can be drawn through them, each line being drawn through two points?

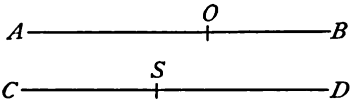
---

[*] NOTE.—The pupil is expected to study the suggestions carefully, to follow the directions when directions are given, to answer the questions when questions are asked, and to give the authority on which the answers are based; then to review the whole demonstration in a consecutive manner, without the aid of the suggestions. To illustrate what is expected of the pupil, model demonstrations are given of a few propositions, but no model should be consulted until after the proposition has been studied by means of the suggestions. The pupil will be more likely to avoid indefiniteness if he writes out all of his demonstrations for the first few weeks.

Model.

PROPOSITION I.

Theorem. *Two straight angles are equal.*



Let A O B and C S D represent two straight angles.

To prove that angle A O B is equal to angle C S D.

The two sides of $\angle A O B$ and $\angle C S D$ form straight lines. § 18.

Place $\angle A O B$ upon $\angle C S D$ so that point O lies upon S and some other point of line $A O B$, as A, lies upon a part of line $C S D$. § 40, P. 2.

The lines $A O B$ and $C S D$ coincide, and therefore are one and the same line. *Ax.* 12.

$\therefore$ The $\angle A O B = \angle C S D$. *Ax.* 13, § 17 c.

Therefore—Two straight angles are equal. Q. E. D.

Ex. 2. Given four points, no three of which are in the same straight line, how many straight lines can be drawn through them if each line connects two of the four points?

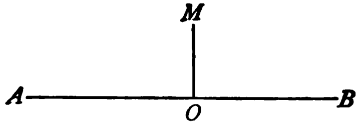
Ex. 3. What is the greatest number of points in which three straight lines can intersect?

Ex. 4. What is the greatest number of points in which four straight lines can intersect?

Ex. 5. If an angle is a right angle, what is its supplement?

PROPOSITION II.

43. Theorem. *One straight angle is equal to two right angles.*



Let A O B represent a straight angle.

To prove that the angular magnitude A O B is equal to two right angles.

Suggestion 1. What kind of a line is *A O B*? § 18.

2. Let *M O* represent a straight line that makes equal angles with the line *A B*. *Ax.* 14.

What kind of ∠s are *A O M* and *M O B*? § 20.

Therefore—

44. A perpendicular is **erected to a line** when it is drawn perpendicular to the line from a point in the line.

45. A perpendicular is **dropped to a line** when it is drawn perpendicular to the line from a point without the line.

Ex. 6. If an angle is two-thirds of a right angle, what is its supplement?

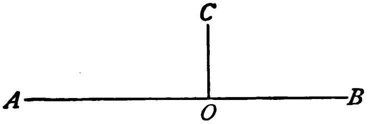
Ex. 7. If an angle is three-fourths of a right angle, what is its complement?

Ex. 8. The complement of angle x equals one-third of its supplement. Find what part of a right angle x is.

Ex. 9. The supplement of angle x is two and one-half times its complement. Find x, the complement, and the supplement, each in terms of a right angle.

PROPOSITION III.

46. Theorem. *At any point in a straight line one perpendicular, and but one, can be erected.*



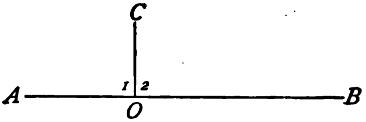
Let A B represent a straight line and O any point in the line.

FIRST.—*To prove that a perpendicular can be erected to A B at O.*

Suggestion 1. Draw *C O* to meet *A B* at *O*, making the two adjacent angles equal. *Ax.* 14.

2. See § 20 and 23.

SECOND.—*To prove that but one perpendicular can be erected to A B at O.*



Suggestion 1. Represent a perpendicular to *A B* at *O*, as *C O*. § 46, Part I.

2. How would $\angle$s *1* and *2* compare, if *C O* should be revolved either way, however little? Why? Ax. 15.

3. $\angle$s *1* and *2* would then be what kind of $\angle$s? Auth.

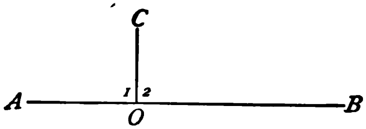
4. *C O* would be what kind of line? § 24.

Therefore—

Model.

PROPOSITION III.

Theorem. *At any point in a straight line one perpendicular and only one can be erected.*



Let A B represent a straight line and O any point in that line.

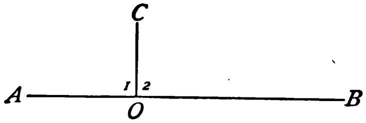
FIRST.—*To prove that a perpendicular can be erected to A B at O.*

Let *C O* represent a line making two equal adjacent ∠s with *A B* at *O*, as ∠ *1* and ∠ *2*. *Ax.* 14.

∠s 1 and 2 are right ∠s. § 20.

C O is ⊥ to *A B*. § 23.

Therefore—At a given point in a line a perpendicular can be erected to the line. Q. E. D.



SECOND.—*To prove that but one perpendicular can be erected to A B at O.*

Let C O represent a perpendicular to A B at O.

If CO is revolved ever so little about the point O, $\angle 1$ would be greater or less than $\angle 2$. *Ax.* 15.

Hence $\angle$s 1 and 2 would be oblique $\angle$s. § 22.

CO would be an oblique line. § 24.

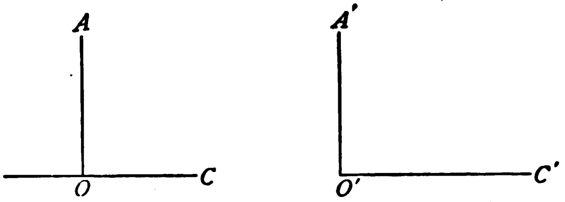
Therefore—Not more than one perpendicular can be erected to a line at a given point. Q. E. D.

47. Corollary.—*Through the vertex of a given angle one and only one straight line which bisects the given angle can be drawn.*

Suggestion.—See method used in demonstrating the theorem. *Axs.* 14 and 15.

Proposition IV.

48. Theorem. ***All right angles are equal.***



Let A O C and A' O' C' represent any two right angles.

To prove that angle AOC and angle $A'O'C'$ are equal.

Suggestion 1. Place $\angle AOC$ upon $\angle A'O'C'$ so that AO lies upon $A'O'$, O upon O'. § 40, P. 2.

2. Where does OC fall? § 46.

3. How then do $\angle AOC$ and $\angle A'O'C'$ compare? *Ax.* 13.

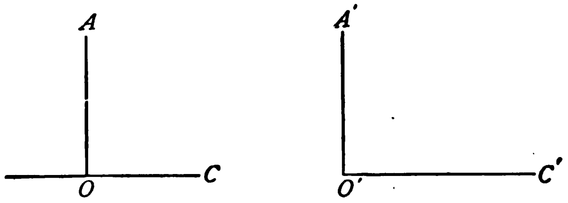
49. Any magnitude is **bisected** when it is divided into two equal parts.

50. Two figures **coincide** when each point of one lies in a corresponding point of the other.

Model.

PROPOSITION IV.

Theorem. *All right angles are equal.*



Let A O C and A' O' C' represent any two right angles.

To prove that angle A O C is equal to angle A'O'C'.

Place ∠ *A O C* upon ∠ *A'O'C'*, *A O* upon *A'O'* with *O* upon *O'*. § 40, P. 2.

Line *O C* falls upon line *O'C'*. § 46.

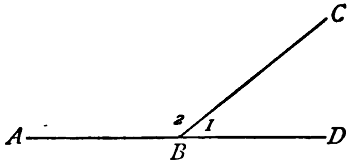
∠ *A O C* coincides with ∠ *A'O'C'*. § 50.

∴ Rt. ∠ *A O C* = Rt. ∠ *A'O'C'*. *Ax.* 13. § 17 (c).

Therefore—All right angles are equal. Q. E. D.

PROPOSITION V.

51. **Theorem.** *If one straight line meets another straight line, the sum of the two adjacent angles formed is equal to two right angles.*



Let C B and A D represent any two straight lines which meet at a point, as B, forming two adjacent angles, as angles 1 and 2.

To prove that the sum of angles 1 and 2 is equal to two right angles.

FIRST.—Suppose ∠s *1* and *2* are equal.

Suggestion 1. How many right angles are formed? § 20.

SECOND.—Suppose ∠s *1* and *2* are unequal.

1. What kind of ∠ is *A B D?* § 18.

2. Compare ∠ *1* + ∠ *2* with ∠ *A B D.* 17 c, *Ax.* 13.

3. See Proposition II. and complete the demonstration.

Therefore—

QUERY.—In Proposition V., what is the hypothesis? What is the conclusion?

52. COROLLARY 1.—*The sum of all the angles on one side of a straight line, having a common vertex in the line, is equal to two right angles.*

53. COROLLARY 2.—*The total angular magnitude about a point is equal to four right angles.*

Ex. 10. If the angular magnitude about a point is divided into six equal angles, each angle is what part of a right angle?

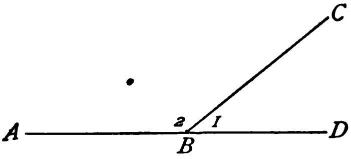
Ex. 11. If the angular magnitude about a point is divided into three angles, the second of which is twice the first, and the third is three times the first, how many right angles in each of the three angles?

Ex. 12. A line drawn perpendicular to the bisector of an angle at the vertex makes equal angles with the sides of the angle.

Model.

PROPOSITION V.

Theorem. *If one straight line meets another straight line, the sum of the two adjacent ángles formed is equal to two right angles.*



Let C B meet A D, at B.

To prove that the sum of angles 1 and 2 is equal to two right angles.

FIRST.—If $\angle$s 1 and 2 are equal.

$\angle 1 + \angle 2 = 2$ Rt. $\angle$s. § 20.

SECOND.—If $\angle$s 1 and 2 are unequal.

$\angle 1 + \angle 2 =$ the st. $\angle A\,B\,D$. § 17 (c), *Ax.* 13.

The st. $\angle A\ B\ D = 2$ Rt. $\angle$s. § 43.

$\therefore\ \angle 1 + \angle 2 = 2$ Rt. $\angle$s. *Ax.* 1.

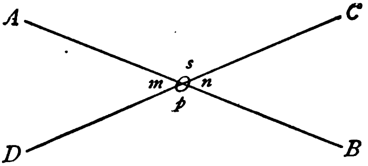
Therefore—If one straight line meets another straight line, the sum of the two adjacent angles is equal to two right angles.

Ex. 13. In Fig. 12, if angle *C O B* is one-half of a right angle, angle *A O B* equals what?

Ex. 14. In Fig. 14, if angle *1* is one-third of a right angle, and angle *4* is three-fourths of a right angle, angle *C B E* equals what?

Proposition VI.

54. **Theorem.** *If two straight lines intersect, the vertical angles formed are equal.*



Let A B and C D intersect at O, forming the vertical angles *m* and *n*, and *s* and *p*.

To prove that angle m is equal to angle n.

Suggestion 1. $\angle m + \angle s =$ what? Why?

2. $\angle s + \angle n =$ what? Why?

3. Compare $\angle m + \angle s$ and $\angle n + \angle s$. Give auth.

4. Compare $\angle m$ and $\angle n$. *Ax.* 3.

Therefore—

In a similar manner compare $\angle s$ and $\angle p$.

Ex. 15. If there are three angles about a point, and one of them is equal to one and one-fifth right angles, and one to nine-tenths of a right angle, what is the magnitude of the other?

Note.—The student should make a careful study of the form and nature of a demonstration. In respect to form: first in order is the statement of the theorem or the **general enunciation**, this should be followed by the application of the theorem to a figure or the **special enunciation**, then follows the **proof**, and finally the **conclusion**.

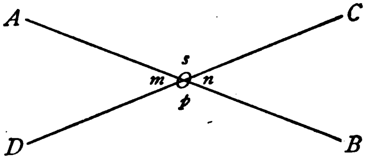
In the special enunciation each feature of the theorem should be carefully applied, close attention being paid, first, to what is given or known, and second, to what is to be determined. In the proof, each statement made should be based upon authority, which should consist of an axiom, a postulate, a definition, or a previously proved proposition. The student should stand ready to demonstrate all propositions used as authority. Care should be taken to see that these authorities exactly apply.

The order of procedure in an original demonstration depends upon the use that can be made of what is given, in arriving at what is to be determined.

Model.

PROPOSITION VI.

Theorem. *If two straight lines intersect, the vertical angles formed are equal.*



Let AB and CD intersect at O, forming the vertical angles *m* and *n*.

To prove that angle m equals angle n.

$\angle m + \angle s = 2$ Rt. $\angle$s. § 51.

$\angle n + \angle s = 2$ Rt. $\angle$s. § 51.

$\therefore \angle m + \angle s = \angle n + \angle s$. *Ax.* 1.

$\therefore \angle m = \angle n$. *Ax.* 3.

QUERY.—In this proposition, which is the special enunciation? Which the general? Which the hypothesis? Which the conclusion?

Polygons.

55. A **polygon** is a portion of a plane bounded by straight lines; as *M N O*, etc., in Fig. 1.

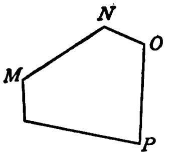


FIG. 1.

56. The bounding lines are the **sides** of the polygon, and their sum is the **perimeter** of the polygon.

57. The angles formed by the sides of the polygon on the side of the inclosed space, are the **interior angles** of the polygon, as angle *B A E* in Fig. 2.

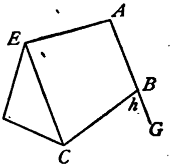


FIG. 2.

58. An angle formed by one side of the polygon and an adjacent side extended, is **an exterior angle** of the polygon; as angle *h* in Fig. 2.

59. The vertices of the interior angles of a polygon are the **vertices of the polygon.**

60. A straight line joining any two vertices, not adjacent, is a **diagonal** of a polygon; as *E C* in Fig. 2.

61. If there is no ambiguity, a polygon may be **read** by naming any two vertices not adjacent; as *A D*, *B D*, *M P*, etc.

62. Polygons are classified according to the **number of their sides.** The least number of sides a polgyon can have is three.

63. A polygon of three sides is a **triangle.**

64. A polygon of four sides is a **quadrilateral.**

65. A polygon of five sides is a **pentagon.**

66. A polygon of six sides is a **hexagon,** etc.

67. An **equilateral polygon** is a polygon all of whose sides are equal.

68. An **equiangular polygon** is a polygon all of whose angles are equal.

69. A **convex polygon** is a polygon no side of which, if extended, enters the space inclosed by the perimeter of the polygon; as *A B C*, etc., Fig. 2.

70. A **concave polygon** is a polygon, two or more sides of which, if extended, would enter the space inclosed by the perimeter of the polygon; as Fig. 3. If either *A B* or *B C* is extended through *B* it would enter the space inclosed by the perimeter of the polygon.

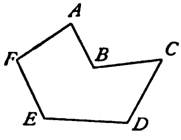


FIG. 3.

71. The angle $A\,B\,C$ in this figure is a **re-entrant angle.**

72. A **right angled triangle,** or a **right triangle,** is a triangle one of whose angles is a right angle.

(a) The side opposite the right angle is the **hypotenuse.**

(b) The other two sides are **the legs** of the triangle.

73. An **acute angled triangle,** or an **acute triangle,** is a triangle all of whose angles are acute.

74. An **obtuse angled triangle,** or an **obtuse triangle,** is a triangle one of whose angles is obtuse.

75. Acute or obtuse triangles are sometimes called **oblique triangles.**

76. A **scalene triangle** is a triangle no two sides of which are equal.

77. An **isosceles triangle** is a triangle which has two equal sides. The equal sides are the **legs** of the triangle.

78. A triangle which has all three sides equal is an **equilateral triangle.**

79. The **base** of a triangle is a selected side or the side upon which it is supposed to stand. The angle which is opposite the base is the **vertical angle** or the **vertex** of the triangle. Generally, any side may be taken as the base, but in an isosceles triangle that side which is not one of the two equal sides is always considered the base.

80. The **altitude** of a triangle is the perpendicular from the vertex to the base or the base extended.

NOTE.—The word altitude may refer to the line or to the length of the line, expressed in terms of some unit. The context will determine which use is intended.

81. The **median line,** or **median** of a triangle, is a line drawn from the vertex to the middle of the opposite side. $A\,B$ is the median, in Fig. 4.

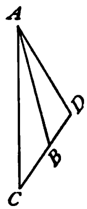
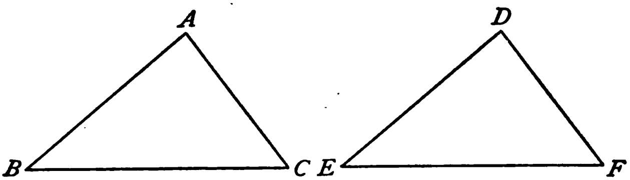


FIG. 4.

PROPOSITION VII.

82. Theorem. *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*



Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.

To prove that triangles A B C and D E F are equal in all respects.

Suggestion 1. Place △ $A\ B\ C$ upon △ $D\ E\ F$, so that point B is upon E and line $B\ C$ lies in $E\ F$. § 40 P.

Where does point C fall? Why?

2. What direction does $B\ A$ take with respect to $E\ D$? § 17 (e).

3. Where does the point A lie? Why?

4. Points C and A being located, where, with respect to $D\ F$, does the line $A\ C$ lie? *Ax.* 12, Cor. 11.

5. What, now, is the position of △ $A\ B\ C$ with respect to △ $D\ E\ F$? § 50.

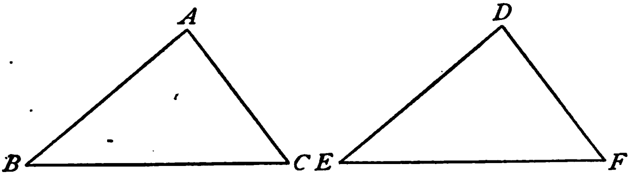
6. How, then, does △ $A\ B\ C$ compare with △ $D\ E\ F$? *Ax.* 13.

Therefore—

Model.

PROPOSITION VII.

Theorem. *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*



Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.

To prove that triangles $A\ B\ C$ and $D\ E\ F$ are equal in all respects.

Place $\triangle\ A\ B\ C$ upon $\triangle\ D\ E\ F$, so that B is upon E and $B\ C$ lies in $E\ F$. § 40, P. 1.

C falls upon F. $(B\ C = E\ F.)$

$B\ A$ takes the direction of $E\ D$. $(\angle B = \angle E.)$ § 17 (e).

A falls upon D. $(B\ A = E\ D.)$

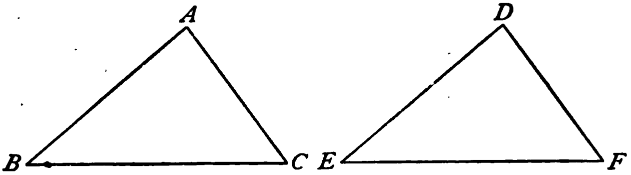
$A\ C$ coincides with $D\ F$. *Ax.* 12 Cor. 2.

$\therefore \triangle\ A\ B\ C = \triangle\ D\ E\ F$. *Ax.* 13. Q. E. D.

NOTE.—The method of proof in this proposition is the method of superposition, and consists in mentally placing one figure upon the other and finding that they exactly coincide.

PROPOSITION VIII.

83. Theorem. *If two triangles have two angles and the included side of one, equal to two angles and the included side of the other, each to each, the triangles are equal in all respects.*



Let ABC and DEF represent two triangles, having BC equal to EF, angle B equal to angle E, and angle C equal to angle F.

To prove that triangles A B C and D E F are equal in all respects.

Suggestion 1. Place △ *A B C* upon △ *D E F*, so that *B* falls upon *E*, and *B C* on *E F*. Where does *C* fall? Why?

2. What direction does *B A* take? Why?

3. Where does *A* fall? Why? Sug. 2.

4. What direction does *C A* take? Why?

5. Where, now, does the point *A* fall? Sugs. 2 and 4.

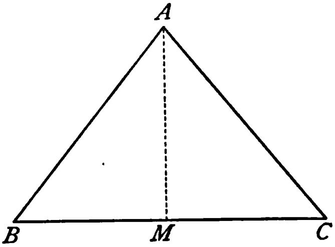
6. Then how does △ *A B C* compare with △ *D E F?* *Ax.* 13.

Therefore —

84. Scholium. In equal triangles, equal angles lie opposite equal sides, and equal sides lie opposite equal angles.

PROPOSITION IX.

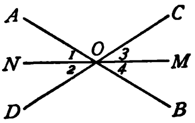
85. Theorem. *The angles opposite the equal sides of an isosceles triangle are equal.*



Let ABC represent an isosceles triangle, AB being equal to AC.

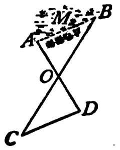
To prove that angle B is equal to angle C.

Suggestion 1. Let $A M$ be drawn to represent a bisector of $\angle A$, and be extended until it meets $B C$, as at M.
2. Compare $\triangle A B M$ with $\triangle A C M$. § 82.
3. Compare $\angle B$ with $\angle C$. § 84.
Therefore —



Ex. 16. If a straight line bisects one of a pair of vertical angles, prove that it bisects the other also.

If $M N$ bisects $\angle A O D$, prove that it bisects $\angle C O B$; that is, that $\angle 3 = \angle 4$.

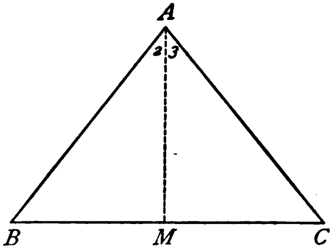


Ex. 17. Let M represent a swamp or pond. Required to find the distance $A B$. How can this be done?

Model.

PROPOSITION IX.

Theorem. *The angles opposite the equal sides of an isosceles triangle are equal.*



Let ABC represent an isosceles triangle, AB being equal to AC.

To prove that angle B is equal to angle C.

Let AM be drawn to represent the bisector of $\angle A$, and be extended to meet BC, as at M.

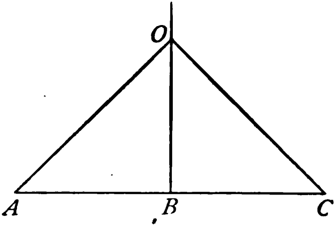
In $\triangle$s ABM and ACM,

$AB = AC.$	Hyp.
$\angle 2 = \angle 3.$	Cons.
$AM = AM.$	*Ax.* 13.
$\therefore \triangle ABM = \triangle ACM.$	§ 82.
Therefore: $\angle B = \angle C.$	§ 84.
	Q. E. D.

Ex. 18. If the equal sides of an isosceles triangle are extended beyond the base, prove that the exterior angles formed with the base are equal.

PROPOSITION X.

86. Theorem. *If a perpendicular be erected at the middle point of a straight line, the distances from any point in the perpendicular to the extremities of the line are equal.*



Let A C represent any line, B its middle point, B O a perpendicular to A C at B, and O any point in the perpendicular.

To prove that O A is equal to O C.

Suggestion 1. In △s *O B A* and *O B C*, what parts are equal, each to each? Why?

2. How do △s *O B A* and *O B C* compare? § 82.

3. How, then, do *O A* and *O C* compare? § 84.

Therefore —

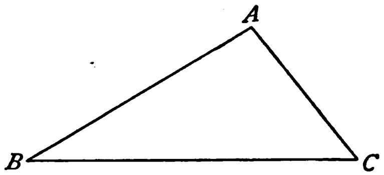
Ex. 19. Points, in the sides of an isosceles triangle, equidistant from the extremities of the base, are equidistant from the vertex.

Ex. 20. Prove that the line which bisects the vertical angle of an isosceles triangle bisects the triangle.

Ex. 21. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base. §s 23 and 20.

PROPOSITION XI.

87. Theorem. *Any side of a triangle is less than the sum of the other two.*



Let A B C represent any triangle.

To prove that any side, as A B, is less than the sum of the other two.

Suggestion: Which represents the shorter distance from *A* to *B*, that by way of the line *A B*, or that by way of the lines *A C* and *C B*? Why? *Ax.* 11.

Therefore —

Ex. 22. If two straight lines intersect and one of the angles formed is a right angle, all of the angles are right angles.

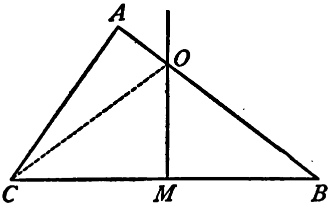
Ex. 23. A line which is perpendicular to the bisector of an angle makes equal angles with the sides of the angle. (1) if drawn through the vertex (Ex. 12); (2) if drawn through any other point of the bisector of the angle.

Ex. 24. Prove that an equilateral triangle is also equiangular.

Ex. 25. If the middle points of the sides of an isosceles triangle be joined by straight lines another isosceles triangle is formed. § 82.

PROPOSITION XII.

88. Theorem. *If a perpendicular be erected at the middle point of a straight line, the distances from a point not in the perpendicular to the extremities of the line are unequal.*



Let B C represent any straight line, M its middle point, O M a perpendicular to B C at the point M, A any point not in the perpendicular, and A B and A C lines drawn from A to the extremities of the line B C.

To prove that A B and A C are unequal.

Suggestion 1. Let O be the intersection of $A B$, and the perpendicular. Draw the line $O C$.

2. Compare $A C$ with $A O + O C$. § 87.

3. Compare $O C$ with $O B$. § 86.

4. Compare $O C + O A$ with $O B + O A$. *Ax.* 2.

5. Compare $A C$ with $A B$. *Ax.* 17.

Therefore —

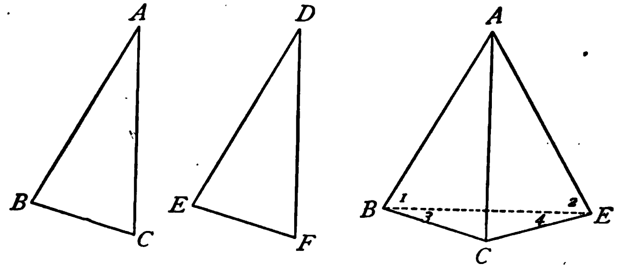
QUERY: Why draw the line $O C$?

Draw the figure so that $A C$ crosses $O M$, and prove the proposition.

Ex. 26. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other. *Ax.* 5.

PROPOSITION XIII.

89. Theorem. *Two triangles, having the three sides of one equal, respectively, to the three sides of the other, are equal in all respects.*



Let ABC and DEF represent two triangles, having AB equal to DE, AC equal to DF, and BC equal to EF.

To prove that triangles ABC and DEF are equal in all respects.

Suggestion 1. Place $\triangle DEF$ upon $\triangle ABC$, so that the longest side, DF, of $\triangle DEF$, coincides with the longest side, AC, of $\triangle ABC$, D upon A, and F upon C, but the point E upon the opposite side of AC, from B. Draw BE.

2. In $\triangle ABE$, compare AB with AE. Give auth.

3. Compare $\angle 1$ with $\angle 2$. § 85.

4. In $\triangle CBE$, compare $\angle 3$ with $\angle 4$. Give auth.

5. Compare $\angle B$ with $\angle E$. *Ax.* 2.

6. Compare $\triangle ABC$ with $\triangle AEC$. § 82.

7. Then how does $\triangle ABC$ compare with $\triangle DEF$?

Therefore —

90. The **premises** of a proposition are the conditions given upon which the conclusion is based.

91. The **converse** of a given proposition is a proposition which has the conclusion of the given proposition for one of the premises, and a premise of the given proposition for the conclusion.

Proposition *XIV.*, separated into premises and conclusion, may be stated:

Premises: The sum of two adjacent angles is equal to two right angles.

Conclusion: The exterior sides form a straight line.

Proposition *XIV.*, is the converse of Proposition *V.*, which, separated into premise and conclusion and interpreted in the language of Proposition *XIV.*, may be stated:

Premises: Two adjacent angles have a straight line for their exterior sides.

Conclusion: The sum of the angles is equal to two right angles.

92. When a proposition is proved to be true, it does not necessarily follow that its converse is also true.

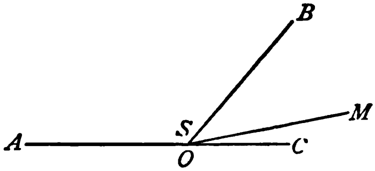
Ex. 27. If $A B C$ is an equilateral triangle, and D, E, and F are points in the sides $A B$, $B C$, and $C A$, respectively, such that $A D$ is equal to $B E$ and to $C F$, prove that triangle $D E F$ is equilateral.

Ex. 28. If the straight line that joins the vertex of a triangle with the middle point of the base is perpendicular to the base, the triangle is isosceles.

Ex. 29. Prove that a line drawn from the vertex of an isosceles triangle to the middle of the base, (*1*) bisects the triangle, (*2*) bisects the vertical angle, (*3*) is perpendicular to the base.

PROPOSITION XIV.

93. **Theorem.** *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*



Let OA, OB, and OC be any three straight lines, which meet to form two adjacent angles S and BOC, whose sum is equal to two right angles.

To prove that the exterior sides, O A and O C, form a straight line.

Suggestion 1. AOC is either a straight line or a broken line. To determine which of these suppositions is true, represent an extension of OA, as OM.

2. How many rt. ∠s in $\angle S + \angle BOM$? Why?

3. How many rt. ∠s in $\angle S + \angle BOC$? Why?

4. Compare the sum of $\angle S + \angle BOM$, with the sum of $\angle S + \angle BOC$. Give auth.

5. Compare $\angle BOC$ with $\angle BOM$. Give auth.

6. Since the equal ∠s BOC and BOM have the common vertex O, and the common side OB, and since OC and OM are on the same side of OB, where does OC lie with respect to OM? Why?

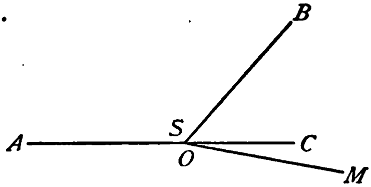
7. OM in an extension of OA. Cons.
What relation does OC sustain to OA?

Therefore —

Model.

Proposition XIV.

Theorem. *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*



Let O A, O B, and O C, be any three straight lines, which meet to form two adjacent angles, S and B O C, whose sum is equal to two right angles.

To prove that the exterior sides, O A and O C, form a straight line.

The line $A\,O\,C$ is either a straight line or a broken line.

Draw $O\,M$ an extension of $O\,A$.

$\angle S + \angle B\,O\,M = 2$ rt. $\angle$s. § 51.

$\angle S + \angle B\,O\,C = 2$ rt. $\angle$s. Hyp.

$\therefore \angle S + \angle B\,O\,M = \angle S + \angle B\,O\,C$. *Ax.* 1.

$\therefore \angle B\,O\,M = \angle B\,O\,C$. *Ax.* 3.

Since the equal $\angle$s $B\,O\,M$ and $B\,O\,C$ have a common vertex and the common side $O\,B$, and since $O\,C$ and $O\,M$ are on the same side of $O\,B$, $O\,C$ must lie upon $O\,M$. § 17 (e).

$A\,O\,M$ is a straight line. Cons.

$O\,C$ lies upon $O\,M$.

$\therefore A\,O\,C$ is a straight line.

Therefore—If the sum of two adjacent angles equals two right angles, their exterior sides form a straight line.

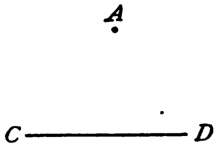
Ex. 30. Two angles are complements of each other, and the greater exceeds the less by 38 degrees. What are the angles?

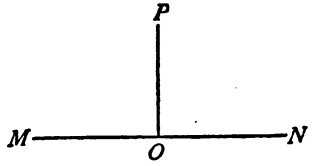
Ex. 31. If two straight lines bisect each other at right angles, any point of either is equidistant from the extremities of the other. § 86.

Ex. 32. In the isosceles triangle *A B C*, *M* and *N* are in the base *B C*, so that angles *B A M* and *C A N* are equal. Prove triangle *B A M* is equal to *C A N*.

PROPOSITION XV.

94. Theorem. *One perpendicular, and only one, can be dropped from a point to a line*





Let C D represent any straight line, and A any point without the line.

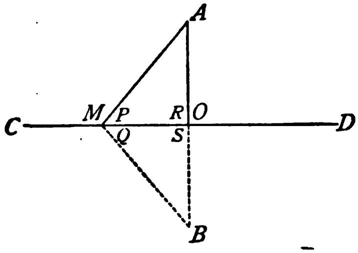
CASE I. *To prove that one perpendicular can be dropped from A to the line C D.*

Suggestion 1. Draw any straight line *M N*, and at any point of this line *O* erect the $\perp$ *O P*. § 46.

2. Place the line *M N* upon the line *C D* and move it back and forth in *C D*. *O P* must at some time embrace point *A*.

Then *O P* is $\perp$ to *C D* from point *A*.

Therefore —



Let A O represent one perpendicular dropped from the point A to the line C D. § 94 Case I.

CASE II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

Suggestion 1. If another ⊥ can be dropped, let it be represented by $A\,M$.

2. Extend $O\,A$ to B, making $O\,B = O\,A$, and connect M and B.

3. Compare $\triangle\, A\,O\,M$ with $\triangle\, B\,O\,M$. Give auth.

4. Compare $\angle P$ with $\angle Q$. Give auth.

5. If by construction $\angle P$ is a rt. ∠, what is $\angle Q$? Is line $A\,M\,B$ a straight or a broken line? Why? § 93.

6. Then how many straight lines are drawn from A to B?

7. What, then, do you conclude about the statement that $A\,M\,B$ is a straight line? Why? *Ax.* 12, Cor. 2.

8. What do you conclude as to the possibility of $A\,M$ being a ⊥ from A to $C\,D$? Why? Sug. 1.

9. Then, how many ⊥s can be dropped from a point to a straight line?

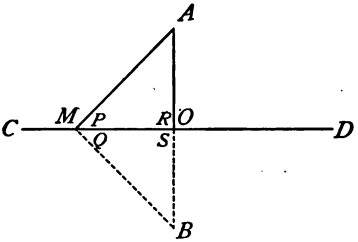
Therefore — ________________

Ex. 33. The supplement of an acute angle is how much more than the complement of the same acute angle?

Model.

PROPOSITION XV.

Theorem. *Only one perpendicular can be dropped from a given point to a line.*



Let O A represent one perpendicular dropped from the point A to the line C D.

CASE II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

If another ⊥ can be dropped let it be represented by $A M$.

Extend $O A$ to B, so that $O A = O B$, and connect M and B.

In △ $A O M$ and $B O M$, $O A = O B$. Cons.
$O M = O M$. Identical.
$\angle R = \angle S$. § 48.
$\therefore \triangle B O M = \triangle A O M$. § 82.
$\therefore \angle P = \angle Q$. § 84.
Since by hypothesis P is a rt. ∠, Q is a rt. ∠. *Ax.* 1.
$\therefore A M B$ is a straight line. § 93.
But $A O B$ is a straight line by construction.
$\therefore A M B$ is a broken line. *Ax.* 12, Cor. 2.
$\therefore A M$ is not ⊥ to $C D$.
∴ But one ⊥ can be dropped from a point to a line.

Q. E. D.

95. It is often convenient to express the magnitude of an angle in some other way than by using a right angle as the unit. To obtain another unit a right angle is divided into ninety equal parts, called **degrees.** The magnitude of an angle may, then, be expressed by stating how many degrees the given angle contains.

96. Theorem. *Complements of equal angles are equal.*

97. Theorem. *Supplements of equal angles are equal.*

NOTE.—In elementary Geometry only acute angles have complements, but either acute or obtuse angles may have supplements.

Ex. 34. If one of two supplementary adjacent angles is bisected, a perpendicular to the bisector through the vertex bisects the other angle.

Ex. 35. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplements of each other.

Ex. 36. How many degrees in a straight angle? In all the angular magnitude about a point?

Ex. 37. How many degrees in the supplement of two-thirds of a right angle?

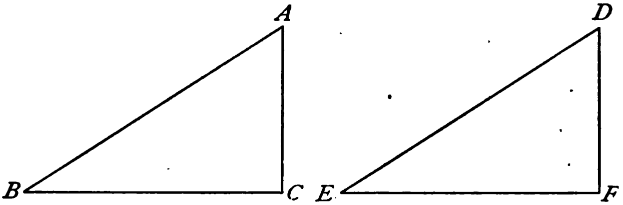
Ex. 38. How many degrees in an angle whose complement equals one-fourth of its supplement?

Ex. 39. The supplement of ten degrees is how much more than the complement of ten degrees?

Ex. 40. The straight lines bisecting the equal angles of an isosceles triangle and terminating in the sides, are equal. § 83.

PROPOSITION XVI.

98. Theorem. *Two right triangles which have the hypotenuse and an adjacent angle of one, equal to the hypotenuse and an adjacent angle of the other, each to each, are equal in all respects.*



Let A B C and D E F represent two right triangles, having the hypotenuse A B equal to the hypotenuse D E, and angle A equal to angle D.

To prove that triangles A B C and D E F are equal in all respects.

Suggestion 1. Place $\triangle\ ABC$ upon $\triangle\ DEF$, so that AB coincides with DE, A upon D, and B upon E.

2. What direction does AC take? Why? § 17 (e).

3. Where does the point C fall? Why?

4. Since BC and EF are both $\perp$ to the line DF, from point E where does BC lie? Why? § 94.

5. Where does point C fall? Why? Sugs. 2 and 4.

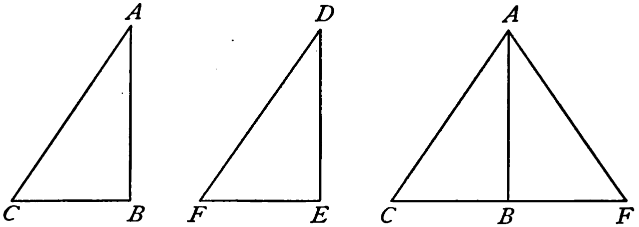
6. How, then, do the two $\triangle$s compare? Why?

Therefore —

NOTE.—In the answer to suggestion 3, it will be seen that C must lie somewhere in the line DF, and, in the answer to Suggestion 5, C must lie somewhere in the line EF. Hence, in the answer to Suggestion 5, C can be exactly located.

PROPOSITION XVII.

99. Theorem. *Two right triangles which have the hypotenuse and a side of one, equal to the hypotenuse and a side of the other, each to each, are equal in all respects.*



Let A B C and D E F represent two right triangles, having the hypotenuse A C equal to the hypotenuse D F, and the side A B equal to the side D E.

To prove that triangles A B C and D E F are equal in all respects.

Suggestion 1. Place the triangles so that *A B* coincides with *D E*, *A* upon *D*, and *B* upon *E*, with the vertices, *C* and *F*, on opposite sides of *A B*.

2. Is *C B F* a straight or broken line? Why? § 93.

3. What kind of an Δ is *A C F?* Why?

4. Compare ∠s *C* and *F*. § 85.

5. Compare Δs *A B C* and *A B F*, *A B C* and *D E F*.

Therefore —

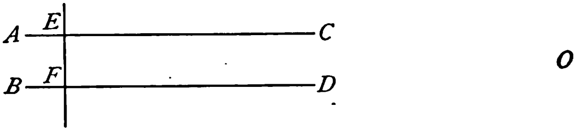
100. Name all the methods of determining the equality of triangles that have been demonstrated. Name those that relate to right triangles.

How many parts of a triangle must be equal in order to make the triangles equal? Suppose the three angles of one triangle equal the three angles of another triangle, each to each, how do the triangles compare in equality?

101. Straight lines, in the same plane, that do not and cannot meet, however far extended, are **parallel lines.**

PROPOSITION XVIII.

102. Theorem. *Two lines, which are perpendicular to the same line, are parallel.*



Let A C and B D represent two lines, each perpendicular to the same line, E F, at the points E and F respectively.

To prove that A C and B D are parallel.

Suggestion 1. *A C* and *B D* either meet or do not meet.

2. If they meet, let *O* represent the point of meeting.

3. Compare the assumption that they meet at *O* with § 94.

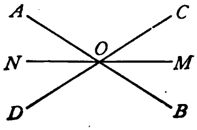
4. Do the lines *A C* and *B D* meet?

5. See § 101.

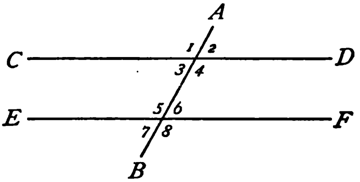
Therefore —

Ex. 41. Through two points an inch apart draw two parallel lines. *Sug.* Use corner of card or sheet of paper.

Ex. 42. If two vertical angles are bisected by two straight lines, prove that the bisectors together form one and the same straight line. Prove that *N O M* is a straight line.



103. A **transversal** or **secant line** is a line which crosses two or more lines; as the line *A B*.



When a transversal cuts two lines eight angles are formed, *viz.*, the angles 1 to 8 in the figure.

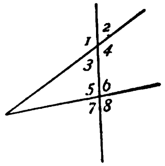
104. The **interior angles** are those within, or between, the lines; as 3, 4, 5 and 6.

105. The **exterior angles** are those without the lines; as 1, 2, 7 and 8.

106. **Alternate interior angles** are pairs of non-adjacent interior angles on opposite sides of the transversal; as 3 and 6, 5 and 4.

107. **Alternate exterior angles** are pairs of non-adjacent exterior angles on opposite sides of the transversal; as 1 and 8, 2 and 7.

108. **Corresponding angles** are pairs of non-adjacent angles on the same side of the transversal, one exterior and one interior; as 2 and 6,4 and 8, etc.

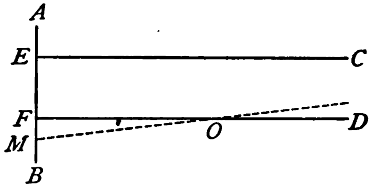


Ex. 43. Draw parallel and transversal lines so as to illustrate all kinds of angles that have been defined.

Ex. 44. If *D* is the middle point of the side *B C*, of triangle *A B C*, and *B E* and *C F* are the perpendiculars from *B* and *C* to *A D*, and *A D* extended, prove that *B E* is equal to *C F*. § 98.

PROPOSITION XIX.

109. Theorem. *If one of two parallel lines is perpendicular to a given line, the other one is perpendicular to the same line.*



Let E C and F D be two parallel lines, and let E C be perpendicular to A B.

To prove that F D is perpendicular to A B.

Suggestion 1. From some point in *F D*, as *O*, draw *O M* to represent a ⊥ to *A B*.

2. What relation does *O M* sustain to *E C?* § 102.

3. What relation does *F D* sustain to *E C?* Hyp.

4. What relation does *M O* sustain to *F D?* *Ax.* 18.

5. Then what relation does *F D* sustain to *A B?* Why? Sug. 1.

Therefore —

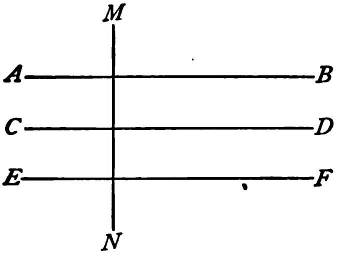
Of what theorem is this the converse?

Ex. 45. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides, are equal.

Ex. 46. If two lines, *A B* and *C D*, intersect in the point *O*, and if the lines *A C* and *B D* be drawn, *A B* + *C D* is greater than *A C* + *B D*.

PROPOSITION XX.

110 Theorem. *If two lines are parallel to the same line, they are parallel to each other.*



Let A B and C D each be parallel to E F.

To prove A B and C D are parallel.

Suggestion 1. Draw $M N \perp$ to $E F$.

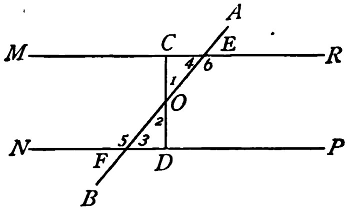
2. How is $M N$ related to $C D$? To $A B$? Why? § 109.

3. How are $A B$ and $C D$ related? Why? § 102.

Therefore —

PROPOSITION XXI.

111. Theorem. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Let M R and N P represent two parallel lines cut by the transversal A B.

CASE I. *To prove that the alternate interior angles 4 and 3 are equal.*

Suggestion 1. Through O, the middle point of EF, draw $CD \perp$ to NP.

2. What relation does CD sustain to MR? Why?

3. Compare $\triangle$s OFD and OEC. Give auth.

4. Then, how do $\angle$s 4 and 3 compare? Why? § 84.

CASE II. *To prove that the alternate interior angles 6 and 5 are equal.*

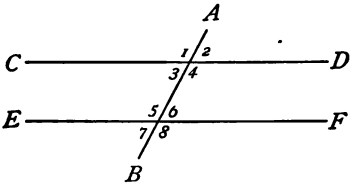
Suggestion 1. $\angle 4 + \angle 6 = \angle 5 + \angle 3$. Why?

2. Compare $\angle$s 6 and 5.

Therefore —

PROPOSITION XXII.

112. Theorem. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



Let C D and E F represent two parallel lines, cut by the transversal AB.

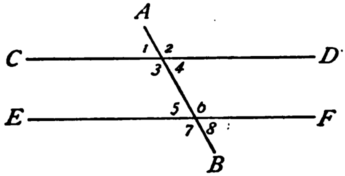
To prove that the corresponding angles 2 and 6, or 3 and 7, etc., are equal.

Suggestion: See § 111.

Ex. 47. In Proposition XIII., place two equal shorter sides upon each other, as AB upon DE, connect C and F, and demonstrate the proposition. *Ax. 3.*

PROPOSITION XXIII.

113. Theorem. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplements of each other.*



Let CD and EF represent two parallel lines cut by the transversal AB, and let 4 and 6 be two interior angles on the same side of AB.

To prove that angles 4 and 6 are supplements of each other.

Suggestion 1. Compare ∠s 3 and 6. Give auth.

2. Compare ∠s 3 and 4. § 28.

3. How, then, does ∠ 6 compare with ∠ 4? Why?

Therefore —

NOTE.—Several different methods of demonstration should be worked out in Propositions *XXII* and *XXIII*.

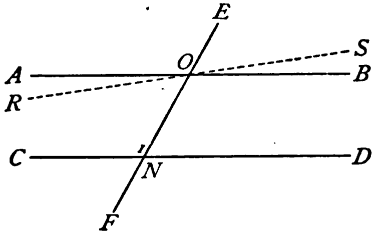
Ex. 48. The line joining the vertices of two isosceles triangles, on opposite sides of the same base, bisects the base and is perpendicular to it.

Suggestion: Method used in § 89.

Ex. 49. If a perpendicular is dropped from the vertex of an isosceles triangle to the base, prove (1) that it bisects the base; (2) that it bisects the vertical angle; and (3) that it bisects the triangle.

Proposition XXIV.

114. Theorem. *If two straight lines are cut by the transversal, so that the alternate interior angles are equal, the lines are parallel.*



Let A B and C D be two straight lines, cut by the transversal E F, so that angles B O F and 1 are equal.

To prove that A B and C D are parallel.

Suggestion 1. Through *O*, draw *R S* to represent a line || to *C D*.

2. Compare ∠ *S O F* with ∠ 1. Give auth.

3. Compare ∠ *B O F* with ∠ 1. Give auth.

4. Compare ∠ *S O F* with ∠ *B O F*. Give auth.

5. Since ∠s *S O F* and *B O F* are superimposed and have one side and the vertex common, what relation must *O S* and *O B* sustain to each other? § 17 (e).

6. Since *R S* is, by construction, || to *C D*, what relation does *A B* sustain to *C D?* Why?

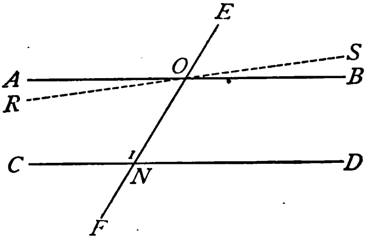
Therefore —

Of what proposition is this the converse?

Ex. 50. If two parallel lines are cut by a transversal, prove that the alternate exterior angles are equal.

Model.

PROPOSITION XXIV.



Theorem. *If two straight lines are cut by a transversal, so that the alternate interior angles are equal, the lines are parallel.*

Let A B and C D be two straight lines, cut by the transversal E F, so that angles B O F and 1 are equal.

To prove that A B and C D are parallel.

Through O, draw $R\,S$ to represent a line ‖ to $C\,D$.
The $\angle S\,O\,F = \angle 1$. § 111.
$\angle B\,O\,F = \angle 1$. Hyp.
$\therefore \angle S\,O\,F = \angle B\,O\,F$. *Ax.* 1.
$\angle S\,O\,F = B\,O\,F$ and $O\,F$ is common, therefore $O\,S$ and $O\,B$ coincide. § 17 (e).
But $O\,S$ is ‖ to $C\,D$, hence $A\,B$ is ‖ to $C\,D$. Q. E. D.

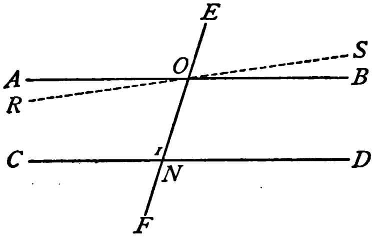
Review.

If parallel lines are cut by a transversal, what pairs of angles are equal?

How may we know lines are parallel? Name two ways.

PROPOSITION XXV.

115. Theorem. *If two straight lines are cut by a transversal so that the corresponding angles are equal, the two straight lines are parallel.*



Let A B and C D represent two straight lines cut by the transversal E F so that the corresponding angles E O A and 1 are equal.

To prove that A B and C D are parallel.

Suggestion 1. Through *O*, draw *R S* to represent a line ‖ to *C D*.

2. Compare ∠ *E O R* with ∠ 1. Give auth.

3. Compare ∠ *E O A* with ∠ 1. Give auth.

4. Compare ∠ *E O R* with ∠ *E O A*. Give auth.

5. What relation does *O A* sustain to *O R*?

6. Since *R S* is, by construction, ‖ to *C D*, what relation does *A B* sustain to *C D*? Why?

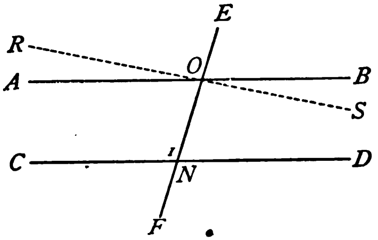
Therefore —

QUERY: If ∠ *1* is a rt. ∠, what previous theorem does *XXV* become?

Ex. 51. In the figure for § 112, compare angles 1 and 6; also angles 4 and 7.

PROPOSITION XXVI.

116. Theorem. *If two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are the supplements of each other, the two straight lines are parallel.*



Let A B and C D represent two straight lines cut by the transversal E F so that angles A O F and E N C are supplements of each other.

To prove that A B and C D are parallel.

Suggestion 1. Through O, draw RS to represent a line ∥ to CD.

2. Compare $\angle AOF$ with $\angle ROF$. (See method in § 115.)

Complete the demonstration.

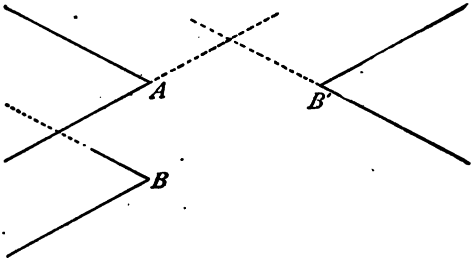
Therefore —

QUERY: If you were to construct a line parallel to a given line, how would you do it?

Ex. 52. In Fig. for § 103, if angle 1 contains 47½ degrees, how many degrees in angle 4? in angle 2? in angle 8? in angle 6? in angle 5? if angle 3 contains 39 degrees, how many degrees in angle 6? in angle 8? in angle 7? in angle 5? Give authority for each statement.

PROPOSITION XXVII.

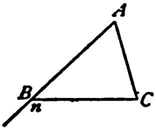
117. Theorem. *Two angles which have their sides respectively parallel and extending in the same direction, or in opposite directions, from their vertices are equal.*



Let A and B represent two angles whose sides are respectively parallel and extend in the same direction from their vertices; and A and B′ two angles whose sides are respectively parallel and extend in opposite directions from their vertices.

To prove angles A and B equal, also angles A and B′.

Suggestion: Extend a side of one angle until it meets a side, or an extended side of the other. Complete the demonstration.



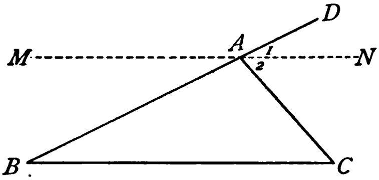
118. Opposite interior angles of a triangle are the two angles of the triangle not adjacent to the exterior angle of the triangle, as $\angle$s A and C. $\angle n$ is an exterior $\angle$. § 58.

Ex. 53. The perimeter of a triangle is less than twice the sum of the medians.

Ex. 54. In what kind of triangle does the bisector of the vertical angle coincide with both the median to the base and the perpendicular from the vertex to the base?

PROPOSITION XXVIII.

119. Theorem. *An exterior angle of a triangle is equal to the sum of the opposite interior angles.*



Let A B C represent a triangle, D A C an exterior angle, and B and C the opposite interior angles.

To prove that angle D A C is equal to the sum of angles B and C.

Suggestion 1. Through the vertex *A* draw a line *M N* ‖ to *B C*.

2. Compare ∠ 1 with ∠ *B*. Give auth.

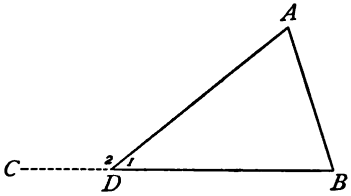
3. Compare ∠ 2 with ∠ *C*. Give auth.

4. Compare ∠ *D A C* with ∠ *B* + ∠ *C*. Give auth.

Therefore —

PROPOSITION XXIX.

120. Theorem. *The sum of the interior angles of a triangle is equal to two right angles*



Let A B D represent a triangle.

To prove that the sum of angles A, B, and 1 is equal to two right angles.

Suggestion 1. Extend one of the sides, as BD.

2. Compare $\angle$s $1 + 2$ with $\angle$s $1 + A + B$.

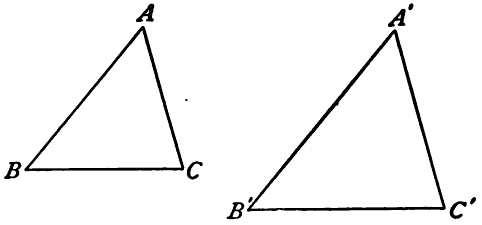
Therefore —

121. COROLLARY I. *A triangle can have only one obtuse angle.*

122. COROLLARY II. *Every right triangle has two acute angles, each of which is the complement of the other.*

PROPOSITION XXX.

123. Theorem. *If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.*



Let ABC and A′B′C′ represent two triangles, having angles A and A′ equal, also angles B and B′.

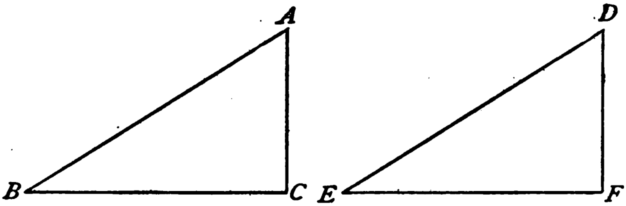
To prove that angles C and C' are equal.

Suggestion: See § 120.

124. COROLLARY. *If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.*

PROPOSITION XXXI.

125. Theorem. *If two right triangles have a side and the opposite angle of one equal to the corresponding side and opposite angle of the other respectively, the right triangles are equal in all respects.*

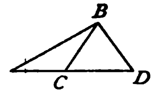


Let ABC and DEF represent two right triangles, in which AC is equal to DF, and angle B is equal to angle E.

To prove that triangles A B C and D E F are equal in all respects.

Suggestion: Compare ∠s *A* and *D*. § 123.
Complete the demonstration.
Therefore —

Ex. 55. Two triangles having two sides and an angle opposite one of them equal each to each may or may not be equal.

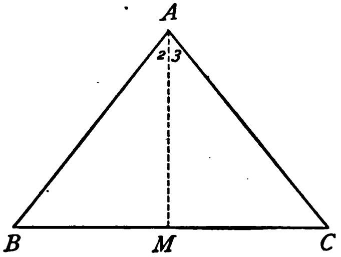


Suggestion: If the equal ∠s are acute and opposite the shorter of the two equal sides, the △s may not be equal.

Study the figure and show this to be true. Show the equality of the △s in other cases. $BD = BC$.

Proposition XXXII.

126. Theorem. *If two angles of a triangle are equal, the sides opposite them are equal and the triangle is isosceles.*



Let A B C represent a triangle in which the angle B is equal to angle C.

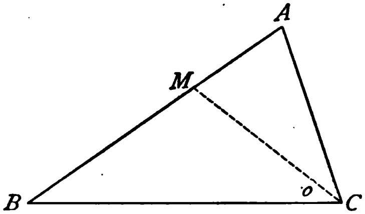
To prove that the side A C is equal to the side A B.

Suggestion 1. Drop a ⊥ from *A* to *B C*, as *A M*.
2. Compare △s *A M B* and *A M C*. Give auth.
3. Compare *A C* with *A B*.
Therefore —

Ex. 56. If *A B C* is a right triangle with the right angle at *C*, and if through *C* a line meeting the hypotenuse at *D* is drawn in such a manner that angle *A C D* equals angle *A*, prove (1)that *C D* bisects the hypotenuse *A B*, and (2) that if angle *A C D* is not equal to *A*, *C D* does not bisect the hypotenuse.

PROPOSITION XXXIII.

127. Theorem. *If two angles of a triangle are unequal, the sides opposite them are unequal, and the greater side is opposite the greater angle.*



Let ABC represent a triangle in which the angle C is greater than the angle B.

To prove that the side B A is greater than the side C A.

Suggestion 1. Draw CM to represent a line making $\angle o$ equal to $\angle B$. Is this possible?

2. Compare BM and CM. Give auth.

3. Compare $CM + MA$ with CA, BA with CA. Give auth.

Therefore —

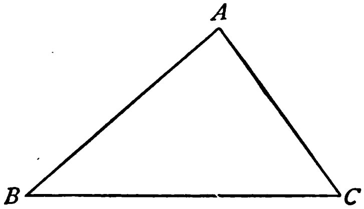
Ex. 57. Prove Proposition XXIX by other methods.

Suggestion: Through A draw a line ‖ to DB. Or extend DA and BA through A, and draw through A a line ‖ to DB.

Ex. 58. If ABC is a right triangle with the right angle at C, and if through C a line meeting the hypotenuse at D is drawn in such a manner that angle ACD equals angle B, prove that CD is perpendicular to the hypotenuse AB.

PROPOSITION XXXIV.

128. Theorem. *If two sides of a triangle are unequal the angles opposite them are unequal, and the greater angle is opposite the greater side.*



Let A B C represent a triangle in which the side A B is greater than the side A C.

To prove that angle C is greater than angle B.

Suggestion 1. $\angle C$ equals $\angle B$, is less than $\angle B$, or is greater than $\angle B$.

2. If $\angle C = \angle B$, compare AB and AC. Can $\angle C = \angle B$? § 126.

3. If $\angle$ C is greater than $\angle B$, compare AB and AC. § 127.

Can $\angle$ C be greater than $\angle$ B?

4. If $\angle C$ is less than $\angle B$, compare AB and AC.

Can $\angle C$ be less than $\angle B$?

Therefore —

Of what proposition is this the converse?

Compare propositions *IX*, *XXXII*, *XXXIII*, and *XXXIV*.

Ex. 59. If from any point in the base of an isosceles triangle perpendiculars are dropped to the sides, these perpendiculars make equal angles with the base.

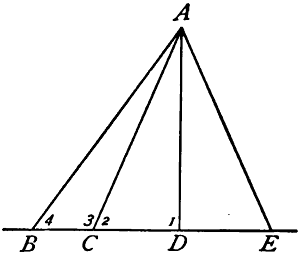
PROPOSITION XXXV.

129. Theorem. *If a perpendicular and oblique lines are drawn from a point to a given line:*

CASE I. *The perpendicular is shorter than any oblique line.*

CASE II. *Two oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.*

CASE III. *Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Let A D be perpendicular to a given line B E, and A B, A C and A E oblique lines, meeting the given line at B, C and E respectively; and let D C be equal to D E and D B greater than D E.

CASE I. *To prove that A D is shorter than any oblique line from A to B E.*

Suggestion 1. How does ∠ 1 compare with ∠ 2? Why?

2. How does *A D* compare with *A C*? Why?

Therefore —

Case II. *To prove that A C equals A E.*

Suggestion 1. Compare triangles *A D C* and *A D E.*

2. Compare *A C* and *A E.* Give auth.

Therefore —

Case III. *To prove that A B is greater than A C or A E.*

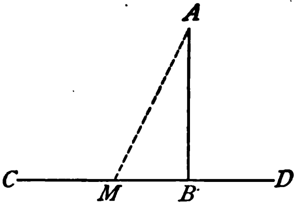
Suggestion 1. Compare ∠s 3 and 2; ∠s 3 and 4.

2. Compare lines *A B* and *A C.* Give auth.

Therefore —

Proposition XXXVI.

130. Theorem. *The shortest line from a point to a straight line is a perpendicular from the point to the line.*



Let A B represent the shortest line from A to the line C D.

To prove that A B is perpendicular to C D.

Suggestion 1. If *A B* is not ⊥ to *C D,* draw *A M* to represent a ⊥ from *A* to *C D.*

2. ∴ *A M* is the shortest line from *A* to *C D.* Why?
§ 129, Case I.

3. What relation does *A M* sustain to *C D? Ax.* 16, Hyp.

4. What relation does *A B* sustain to *C D?*

Therefore —

Of what theorem is this the converse? Compare them.

131. The **distance from a point to a line** is the length of the perpendicular from the point to the line.

132. Propositions *XXXIV* and *XXXVI* are good illustrations of what is known as the indirect method, or the ***reductio ad absurdum*** method of reasoning. Its peculiarity consists in the fact that the statement of the proposition is not directly proved to be true, but that everything which contradicts the statement of the proposition is shown to lead to some manifest absurdity, and is therefore false. This method often presents difficulties to the beginner on account of the fact that he is obliged to admit, temporarily, and for argument's sake, something which the argument itself goes to destroy. The indirect method is employed in demonstration of the converse of propositions. In applying this method care must be taken that every possible case which contradicts the proposition is considered, and each one shown to lead to an absurdity. Then, and then only, is this method of demonstration rigid.

NOTE.—If the student, in original investigation of the above propositions, or any similar one, should chance to consider the supposition which leads to the truth before one or more of the others, the remaining suppositions should be investigated. When all possible suppositions have been stated, one of which is true, and it has been shown that all but one are false, it is evident that the one remaining must be true.

Ex. 60. The line joining the feet of the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides, is parallel to the base.

Ex. 61. The base of an isosceles triangle, together with the bisectors of the angles at the base, forms a second isosceles triangle.

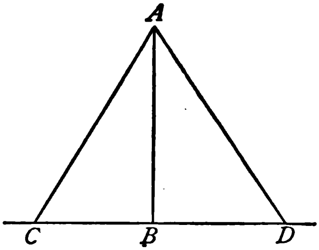
Ex. 62. Two angles having their sides respectively parallel, and each angle, having one side extending in the same direction from their vertices and the other side in opposite directions, are supplements of each other.

Ex. 63. If two triangles have two angles and a side opposite one of them equal respectively to two angles and a corresponding side of the other, the triangles are equal in all respects.

Ex. 64. Two isosceles triangles having equal bases and equal vertical angles are equal.

PROPOSITION XXXVII.

133. Theorem. *Two equal oblique lines, drawn from the same point in a perpendicular to a given line, cut off equal distances from the foot of the perpendicular.*



Let A B represent a perpendicular to C D, and let A C and A D represent equal oblique lines drawn from A to C D.

To prove that B C is equal to B D.

Of what theorem is this the converse?

Therefore—

NOTE.—The pupil should try frequently to demonstrate a proposition with no drawing to assist him. If in recitation he can construct the figure in his mind and make clear his thought to the class, he is worthy of extra credit. Try it with proposition *XXXVII.*

QUERIES —

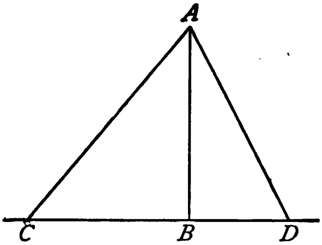
1. By how many methods can we determine that two lines are parallel? §§ 101, 102, 114, etc.

2. By how many methods can we know a line is perpendicular to a line? §§ 23, 109, Ex. 48.

Ex. 65. M and N are two parallel lines. Line A is a perpendicular to M and line B is perpendicular to N. Prove that A and B are parallel.

PROPOSITION XXXVIII.

134. Theorem. *Two unequal oblique lines, drawn from the same point in the perpendicular to a given line, cut off unequal distances from the foot of the perpendicular, the longer line cutting off the greater distance.*



Let AB represent a perpendicular to CD, and AC and AD unequal oblique lines drawn from A to CD, AC being longer than AD.

To prove that BC *is greater than* BD.

Suggestion 1. $BC = BD$, is less than BD, or is greater than BD.

2. If $BC = BD$, how would AC compare with AD? Why?

3. If BC is less than BD, how would AC compare with AD? Give auth.

4. How, then, must BC compare with BD?

Therefore — § 132.

Of what theorem is this the converse?

Ex. 66. Each angle of an equilateral triangle is one-third of two right angles, or two-thirds of one right angle.

Locus of a Point.

135. To locate definitely a point in a plane, two conditions limiting its position must be known.

If only one condition is given, the point is to some extent, but not completely, determined. For example, if a point is at a given distance from some fixed point, it is not exactly located, but may move, provided that in its movement it always satisfies the requirement of being at a given distance from the fixed point. The moving point may occupy any position whatever in a line, which will later be defined as the circumference of a circle with the fixed point as the center.

136. When the position of a point in a plane is limited to and may be anywhere in a line or group of lines, the line or group of lines is the **locus of the point.**

The locus of a point is both inclusive and exclusive. It must include all the points that satisfy the given condition and exclude all that do not satisfy that condition. For example, if one would know that a circumference or any other line or group of lines is a locus, one must ascertain two things: first, that the point **may be anywhere** in the line or set of lines; second, that the point **must be somewhere** in the line or set of lines; i. e., it cannot be anywhere outside of them.

Ex. 67. A straight line drawn from any point in the bisector of an angle to either side and parallel to the other side, makes, with the bisector and the side to which the line is drawn, an isosceles triangle.

PROPOSITION XXXIX.

137. Problem. *To determine the locus of a point at equal distances from the extremities of a given line.*

Let A B represent a given line.

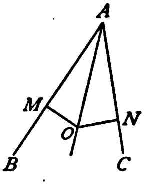
To determine the locus of a point at equal distances from A and B.

Suggestion 1. The problem means that we are to find one or more lines, such that, *first*, any point in them is equally distant from *A* and *B*, and *second*, that no point without them is equally distant from *A* and *B*.

2. What line is everywhere equally distant from the extremities of a given line? § 86.

3. Compare the distances of any points without the ⊥ from the extremities. § 88.

4. What is the required locus?



Ex. 68. Prove that every point in the bisector of an angle is equally distant from the sides of the angle.

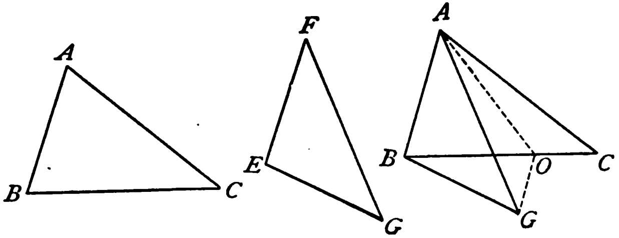
Suggestion: O M and *O N* are ⊥ to *A B* and *A C* respectively. § 131.

Prove *O M* equal to *O N*.

Ex. 69. If a line intersects the sides of an isosceles triangle at equal distances from the vertex, the line is parallel to the base.

PROPOSITION XL.

138. Theorem. *If two triangles have two sides of one equal to two sides of the other and the included angles unequal, the remaining sides are unequal, and that side is the greater which is opposite the greater included angle.*



Let ABC and FEG represent two triangles in which AB is equal to FE, AC is equal to FG, and angle A is greater than angle F.

To prove that BC is greater than EG.

Suggestion 1. Place $\triangle FEG$ upon $\triangle ABC$, so that FE coincides with AB, F upon A, and E upon B.

2. Since $\angle F$ is less than $\angle A$, where does FG lie with respect to $\angle A$? § 17 (e).

3. Bisect $\angle GAC$ and extend to meet BC, as at O. Connect O and G.

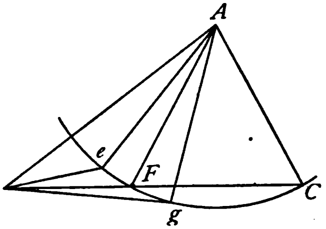
4. Compare $\triangle$s AOG and AOC.

5. Compare lines OG and OC.

6. Compare BG with $BO + OG$; BG with BC.

Therefore —

In placing △ FEG upon △ ABC, G may be without the △ ABC, as at g, upon line BC, as at F, or within △ ABC, as at e, as illustrated by the accompanying diagram.



Apply the demonstration above to the figure, with G in each of the three positions.

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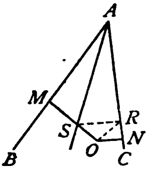
70. Prove that every point not in the bisector of an angle is unequally distant from the sides of the angle.



Let O be at any point not in the bisector of the angle.

Prove that OM and ON are unequal.

Suggestion 1. Draw $SR \perp$ to AN.

2. Connect R and O.

3. Compare $SO + SR$ with OR, OM with OR, OM with ON.

Ex. 71. What is the locus of a point equally distant from the sides of an angle? Exs. 68 and 70.

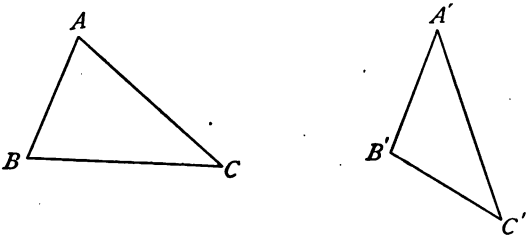
Ex. 72. What is the locus of a point at equal distances from the two intersecting lines? Ex. 71.

Ex. 73. The middle point of the hypotenuse of a right triangle is at equal distances from the three vertices.

Suggestion: Ex. 56, or drop ⊥s from the middle point of the hypotenuse to the legs of the △ and compare the triangles formed.

PROPOSITION XLI.

139. Theorem. *If two triangles have two sides of one equal to two sides of the other and the third sides unequal, the angles opposite the third sides are unequal, and that angle is the greater which is opposite the greater side.*



Let ABC and A′B′C′ represent two triangles having AB equal to A′B′, AC equal to A′C′, and BC greater than B′C′.

To prove that angle A is greater than angle A′.

Suggestion 1. If $\angle A = \angle A'$, how do BC and $B'C'$ compare? Why?

2. If $\angle A < \angle A'$, how do BC and $B'C'$ compare? Why?

3. How, then, must $\angle A$ compare with $\angle A'$?

Therefore —

Quadrilaterals.

140. Quadrilaterals are divided into classes, as follows:

(a) The **trapezium,** which has no two sides parallel. Fig. 1.

(b) The **trapezoid,** which has two sides parallel. Fig. 2.

(c) The **parallelogram,** which has its pairs of opposite sides parallel. Fig. 3.

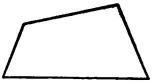


FIG. 1.

141. The parallel sides of a trapezoid are the **bases,** the non-parallel sides are the **legs,** and the line perpendicular to the bases is the **altitude.**

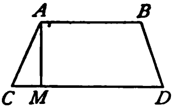


FIG. 2.

In Fig. 2, *A B* and *C D* are the bases, *A C* and *B D* are the legs, and *A M* is the altitude of the trapezoid.

142. Any side of a parallelogram may be selected as the base. It is then called the **primary base.** The side opposite the primary base is the **secondary base.** The other sides of the parallelogram are its **sides or legs.**

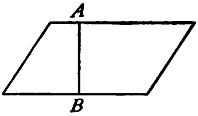


FIG. 3.

143. The **altitude** of a parallelogram is a line perpendicular to the bases, as, *A B* in Fig. 3.

NOTE.—Usually the words **lower base** and **upper base** are used instead of primary base and secondary base, respectively, but as Geometry does not take into account the idea, up and down, the terms primary and secondary are preferable.

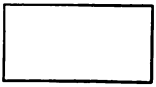


FIG. 4.

144. An oblique-angled parallelogram is a **rhomboid.** Fig. 3.

145. A right-angled parallelogram is a **rectangle.** Fig. 4.

146. An equilateral rhomboid is a **rhombus.** Fig. 5.

147. An equilateral rectangle is a **square.** Fig. 6.

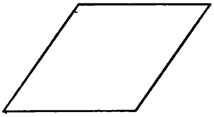


FIG. 5.

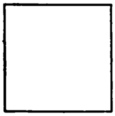
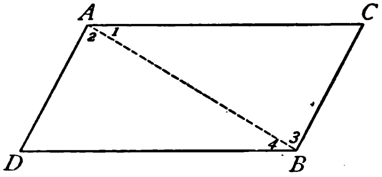


FIG. 6.

The student should make out his own classification of quadrilaterals.

Proposition XLII.

148. Theorem. *The opposite sides and opposite angles of a parallelogram are respectively equal, and adjacent angles are supplements of each other.*



Let A C B D represent a parallelogram.

To prove that A C equals D B; C B equals A D; that angle A and angle B, angle C and angle D are respectively equal; and that the adjacent angles A and C, C and B, B and D, and D and A are respectively supplementary.

Suggestion 1. Draw the diagonal $A\,B$, and compare △s $A\,B\,D$ and $A\,B\,C$.

2. How do $A\,C$ and $B\,D$ compare? $A\,D$ and $C\,B$? Complete the demonstration.

Therefore —

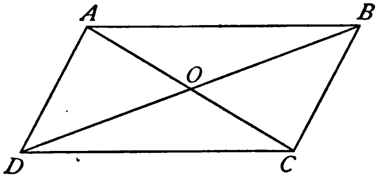
149. Corollary 1. *The diagonal of a parallelogram divides it into two equal triangles.*

Ex. 74. The sum of the exterior angles at the base of any triangle is equal to two right angles plus the vertical angle of the triangle.

Ex. 75. In a triangle $A\,B\,C$, the angle C is twice the sum of angles A and B, and angle B is twice angle A; find all three angles of the triangle.

PROPOSITION XLIII.

150. Theorem. *The diagonals of a parallelogram bisect each other.*



Let AC represent a parallelogram, and AC and BD the diagonals.

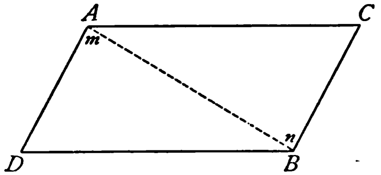
To prove that A C and D B bisect each other.

Suggestion. Prove by comparison of triangles.

Therefore —

PROPOSITION XLIV.

151. Theorem. *If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.*



Let ACBD represent a quadrilateral in which the side A C is equal and parallel to the side B D.

To prove that A C B D is a parallelogram.

Suggestion 1. What is the definition of a ▱?

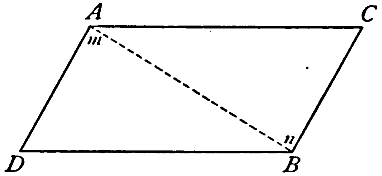
2. How much of the definition is given in the Hyp.? What remains to be proved?

3. Draw the diagonal *A B*. Compare $\triangle$s *A B C* and *A B D*. Compare $\angle m$ and $\angle n$. Complete the demonstration.

Therefore —

PROPOSITION XLV.

152. Theorem. *A quadrilateral whose opposite sides are equal is a parallelogram.*



Let C D represent a quadrilateral in which A C is equal to D B, and A D is equal to C B.

To prove that A C B D is a parallelogram.

Suggestion 1. Draw the diagonal *A B*.

2. Two definitions of a ▱ may now be used.

§§140 (c), 151.

3. How much of either is given in the theorem? What remains to be proved?

Work out a demonstration by use of each definition.

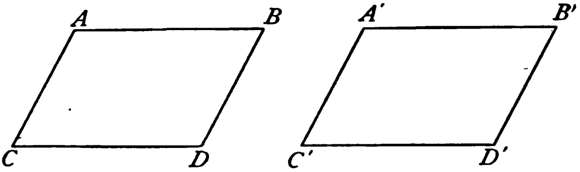
State the different definitions of a parallelogram that have now been established. Commit each to memory.

NOTE.—The pupil should carefully distinguish between a **property** of a figure and a **definition.**

A definition of a figure is such a description of it as serves to distinguish it from every other figure (§ 151). A property of a figure is a statement of it which is true, and may also be true of other figures (§ 148).

PROPOSITION XLVI.

153. Theorem. *Two parallelograms which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal in all respects.*



Let AD and A′D′ represent two parallelograms in which AB is equal to A′B′, AC is equal to A′C′ and angle A is equal to angle A′.

To prove that A D and A′ D′ are equal in all respects.

Suggestion 1. Place ▱ *A D* upon ▱ *A′ D′* so that *A B* coincides with *A′ B′*, *A* upon *A′*, and *B* upon *B′*, and so that the two figures fall upon the same side of *A′ B′*.

2. What direction does *A C* take? Why?

3. Where does the point *C* fall? Why?

4. What direction does *C D* take? Why? *Ax.* 18.

5. Where does the point *D* fall? Why?

Complete the demonstration.

Therefore —

Ex. 76. If a parallelogram has one right angle it is a rectangle.

Ex. 77. A quadrilateral is a parallelogram if its diagonals bisect each other.

Ex. 78. A quadrilateral is a parallelogram if its opposite angles are equal.

PROPOSITION XLVII.

154. Theorem. *The diagonals of a square or rhombus bisect the angles of the square or rhombus.*

Let A B C D represent a square or rhombus, A C and B D the diagonals.

To prove that angles A, B, C, and *D* are bisected.

Suggestion. Try to demonstrate this proposition by constructing the figure in your mind, and thus establish the theorem without the aid of a drawing.

Ex. 79. The diagonals of a square and rhombus intersect at right angles.

Try to find at least three methods by which to prove this exercise.

Ex. 80. Given a square *A D B C*. Draw diagonal *C D*.

Lay side *C B* upon the diagonal, *C* upon *C*, and *B* upon some point, as *E*. At *E* erect a perpendicular and extend to the side *D B*, meeting it at *F*.

Prove that *D E* is equal to *E F* and equal to *F B*.

Suggestion. Connect *C* and *F*.

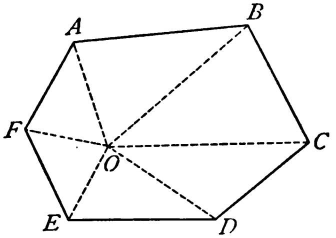
Ex. 81. If the diagonals of a parallelogram intersect at right angles, the figure is a square or rhombus.

Try to prove by at least two methods.

Ex. 82. If a straight line intercepted between parallel lines is bisected, any other straight line drawn through the point of bisection to meet the parallel lines is also bisected at that point. The two intersecting lines intercept equal portions on the parallel lines.

PROPOSITION XLVIII.

155. Theorem. *The sum of the interior angles of any convex polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.*



Let A D represent any convex polygon.

To prove that the sum of the interior angles of the polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.

Suggestion 1. Connect each vertex with *O*, any point within the polygon.

2. If the polygon has *n* sides, how many △s are formed?

3. How many right ∠s in the sum of the interior angles of all of the △s?

4. What is the sum of the ∠s about *O?*

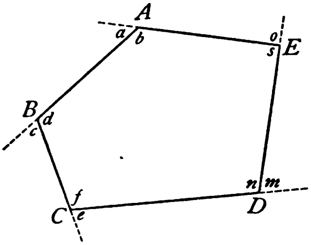
5. Then, how many right ∠s in the sum of the interior ∠s of the polygon?

Therefore —

Ex. 83. Draw all possible diagonals of a polygon from a given vertex, and prove Prop. *XLVIII* by another method.

PROPOSITION XLIX.

156. Theorem. *The sum of the exterior angles of any convex polygon, formed by extending each side through one vertex in order, is equal to four right angles.*



Let AED, etc., represent a convex polygon.

To prove that the sum of the exterior angles a, c, e, etc., is equal to four right angles.

Suggestion 1. What is the sum of ∠s *a* and *b?* Of ∠s *c* and *d?* Of ∠s *f* and *e*, etc.?

2. If the polygon has *n* sides, the sum of all the exterior and interior ∠s equals how many right ∠s?

3. How many right ∠s in ∠ *a* + ∠ *c* + ∠ *e*, etc. *Therefore* —

Ex. 84. How many right angles in the sum of the interior angles of a pentagon? of a triangle? of a hexagon? of an octagon? of a decagon, etc.?

Ex. 85. How many right angles in the interior angles of a concave polygon? Compare with § 155.

Review.

157. Name all the authorities:

1. (*a*) by which two triangles may be determined equal.

(*b*) by which two angles may be determined equal.

(*c*) by which two lines may be determined parallel?

(*d*) by which a line may be known to be perpendicular to another line.

(*e*) by which a quadrilateral may be known to be a parallelogram? a parallelogram to be a rectangle? a parallelogram to be a square or rhombus?

2. Name all the loci thus far demonstrated.

3. How may triangles be known to be isosceles?

4. State the relation that exists between angles of a triangle and sides opposite; between sides and angles opposite.

Ex. 86. What is the locus of a point two inches from a given line indefinite in length? Three inches from the line?

Ex. 87. Find a point x that is four inches from a given line and also in another given line.

Use locus in determining x. Is it possible to find more than one x? Suppose the two lines are parallel and four inches apart; parallel and any other distance apart?

Ex. 88. Find point x if it is in a given line and also equally distant from two intersecting lines. Ex. 72.

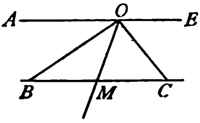
Ex. 89. Find point x if it is equally distant from two intersecting lines and also a given distance from a given line. Ex. 86.

Ex. 90. Find point x if it is equally distant from two intersecting lines, and also equally distant from two given points.

Ex. 91. Find point x if it is equally distant from the extremities of a given line, and also a given distance from a given line.

Ex. 92. The three perpendicular bisectors of the sides of a triangle meet at a point.

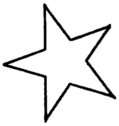
Sug.: Use locus in demonstrating Exs. 90, 91, and 92.



Ex. 93. In the figure, AE and BC are parallel, and MO is a transversal. BO bisects the angle AOM, and CO bisects the angle EOM. Prove that BM is equal to MC.

Ex. 94. If the legs of a trapezoid are equal, they make equal angles with the bases. *Sug.:* Draw through a vertex of the shorter base a line parallel to one of the legs.

Ex. 95. If the legs of a trapezoid make equal angles with the bases the legs are equal.



Ex. 96. The sum of the angles at the vertices of a five-pointed star is equal to two right angles.

Ex. 97. ABC is an isosceles triangle, D any point in the base BC, DM a line parallel to AB, and DN a line parallel to AC, M and N being points in AC and AB respectively. Prove that the perimeter of $AMDN$ is equal to the legs of the triangle.

Ex. 98. A line that embraces the vertex of an angle and a point equidistant from the sides, bisects the angle.

Ex. 99. The difference between any two sides of a triangle is less than the third side.

CHAPTER II.

THE CIRCLE.

Definitions.

158. A **circle** is a portion of a plane bounded by a curved line, all points of which are equally distant from a fixed point within.

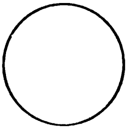


FIG. 1.

159. The fixed point is the **center** of the circle, and the bounding line is the **circumference** of the circle.

NOTE—In higher branches of mathematics the word circle is also used to denote what is here defined as the circumference; that is, the curved *line* bounding a portion of a plane instead of that portion of a plane itself, but in this book the above definitions will be adhered to.

160. A **radius** is any straight line drawn from the center of a circle to its circumference, as *A O*, Fig. 2.

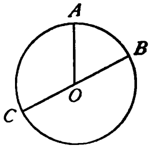


FIG. 2.

161. A **diameter** is any straight line drawn through the center, terminated by the circumference, as *C B*, Fig. 2.

162. COROLLARY.—*From the definition of a circle, all radii of the same circle are equal; also, all diameters of the same circle are equal, each diameter being twice the radius.*

163. An **arc** of a circle is any portion of its circumference, as line *A M B*, Fig. 3.

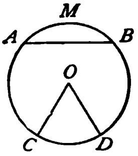


FIG. 3.

164. A **chord** of a circle is a straight line joining any two points of the circumference, as *A B*, Fig. 3.

(*a*) When the extremities of a chord and an arc are the same, the chord **subtends** the arc.

(*b*) In Fig. 3, the chord *A B* subtends the arc *A M B*, and also the arc *A C D B*. Thus, any chord subtends two arcs, which, together, make up the whole circumference. When an arc and its chord are spoken of, the smaller of the two arcs is always understood, unless the other is specifically stated. The terms **major** and **minor arcs** are sometimes used.

165. A **secant** of a circle is any straight line meeting the circumference in two points, and passing through the circumference in at least one of them, as *C D E*, Fig. 4.

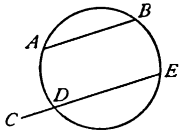


FIG. 4.

A secant is a chord extended.

166. A **tangent** to a circle is a line which touches the circumference at one, and only one, point.

In Fig. 5, the line *M N* is tangent to the circle, and the circle is tangent to the line.

(*a*) The line may be straight or curved. If the curved line is a circumference, a circle is tangent to another circle.

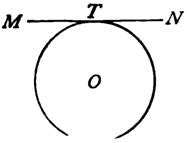


FIG. 5.

(*b*) The point common to the line and the circle or to the two circles is the **point of contact** or **point of tangency.**

167. A **segment** of a circle is a portion of a circle bounded by an arc and its subtending chord, as segment *A M B*, Fig. 3.

168. A **sector** of a circle is a portion of a circle bounded by two radii and the intercepted arc. In Fig. 3, the sector *C O D* is bounded by the arc *C D*, and the radii *O C* and *O D*.

169. The arc intercepted between two radii **subtends** the angle made by the radii; arc $C D$, Fig. 3, subtends angle $C O D$.

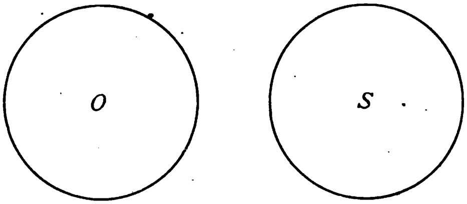
170. An **angle at the center** of a circle is an angle formed by two radii.

171. **Concentric circles** are circles that have the same center.

172. A circle may be **read** by naming the letter at the center of the circle, or by naming two or more letters on the circumference.

PROPOSITION I.

173. Theorem. *Two circles are equal if the radius of one equals the radius of the other.*



Let O and S represent two circles having equal radii.

To prove that circle O and circle S are equal.

Suggestion 1. Place ⊙ O upon ⊙ S, with the center O upon the center S.

2. Where must the circumference of O fall with respect to the circumference of S? Why?

Therefore —

INDIRECT METHOD.

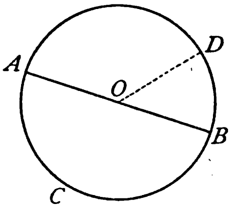
Suggestion 1. Same as above.

2. Suppose some part of the circumference of one ⊙ should fall outside of the circumference of the other ⊙, how would the radii compare?

Therefore—

PROPOSITION II.

174. Theorem. *A diameter divides a circle into two equal parts.*



Let A D B C represent a circle, and A B a diameter.

To prove that A B divides the circle into two equal parts.

Suggestion 1. Let *D* be any point in the arc *A D B*, except *A* and *B*. Connect the center *O* with *D*.

2. Revolve the segment *A D B* upon the line *A O B*, as an axis, into the plane *A C B*.

3. Where does the point *D* fall? § 158.

4. Since *D* is any point in the arc *A D B*, compare the segments *A D B* and *A C B*. *Ax.* 13.

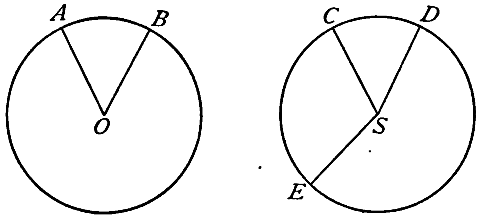
Therefore—

175. COROLLARY.—*A diameter divides the circumference into two equal parts.*

176. A **semicircle** is a segment of a circle bounded by a diameter and the arc it subtends, as *A D B*, Fig., § 174.

PROPOSITION III.

177. Theorem. *In the same circle, or in equal circles, equal angles at the center intercept equal arcs at the circumference, and of two unequal angles, the greater intercepts the greater arc.*



Let O and S represent equal circles, and let A O B and C S D represent equal angles at the centers O and S respectively.

CASE I. *To prove that arc A B is equal to arc C D.*

Suggestion 1. Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *A O* upon *C S*. Where does the circumference of ⊙ *O* fall? Why?
Where does point *A* fall? Why?

2. What direction does *O B* take? Why?

3. What is the location of arc *A B*? Why?

Therefore —

Let angle C S E be greater than angle A O B.

CASE II. *To prove that arc E C is greater than arc A B.*

Suggestion 1. Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *O B* upon *S C*. Where does *O A* fall with respect to ∠*C S E*? Why? § 17 (e).

2. Where does point A fall with respect to arc CE? Why? Compare arc AB and CE.

Therefore —

PROPOSITION IV.

178. **Theorem.** CONVERSELY. *In the same circle, or in equal circles, equal arcs subtend equal angles at the center, and of two unequal arcs, the greater arc subtends the greater angle at the center.*

Let O and S (§ 177) represent equal circles, and A B and C D equal arcs.

CASE I. *To prove that angle O is equal to angle CSD.*

Suggestion. Place ⊙ O upon ⊙ S, the center O upon the center S, so that arcs AB and CD coincide. Why is this possible?

Complete the demonstration.

Therefore —

Let arc E C be greater than arc A B.

CASE II. *To prove angle CSE is greater than angle O.*

Suggestion 1. Place ⊙ O upon ⊙ S, the center O upon the center S, and B upon C. Where does point A fall with respect to arc CE? Why?

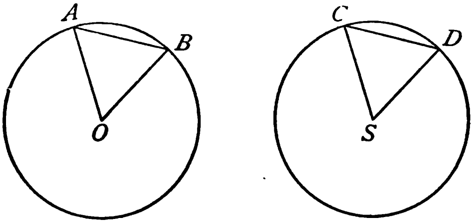
2. Where does line OA fall? Why?

Therefore —

Ex. 100. The locus of a point at a given distance from a given point is the circumference of a circle, drawn with the given point as a center and with the length of the given distance as a radius. § 136.

PROPOSITION V.

179. Theorem. *In the same circle, or in equal circles, chords which subtend equal arcs are equal.*



Let O and S represent equal circles, in which arc A B is equal to arc C D.

To prove that chord $A\ B$ is equal to chord $C\ D$.

Suggestion 1. Draw the radii $O\ A$, $O\ B$, $S\ C$, and $S\ D$.

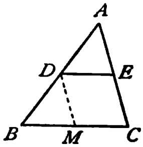
2. Compare $\angle$s O and S. Give auth.

3. Compare Δs $A\ O\ B$ and $C\ S\ D$. Give auth.

4. Compare chord $A\ B$ with chord $C\ D$.

Therefore —

Ex. 101. If a line, parallel to the base of a triangle, bisects one side, it bisects the other side also, and is equal to one-half of the base. $A\ B$ is bisected at D, and $D\ E$ is parallel to $B\ C$.



Prove $A\ E = E\ C$, and $D\ E = \frac{1}{2}\ B\ C$.

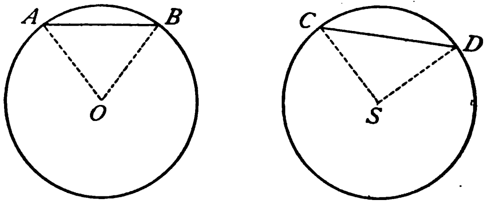
Suggestion. Draw $D\ M \parallel$ to $A\ C$.

Ex. 102. With the same construction as in exercise 101, connect E and M, and prove the exercise.

Ex. 103. If a line bisects the two legs of a triangle, prove that it is parallel to the base, and equals one-half of the base. Direct and indirect methods.

PROPOSITION VI.

180. Theorem. *In the same circle, or in equal circles, two chords which subtend unequal arcs are unequal, that chord being greater which subtends the greater arc.*



Let O and S represent equal circles, and let the arc A B be less than the arc C D.

To prove that the chord A B is less than the chord C D.

Suggestion 1. Connect the extremities of the arcs with their respective centers.

2. Compare $\angle S$ with $\angle O$. § 178.

3. Compare chord $C D$ with chord $A B$. § 138.

Therefore —

Ex. 104. A diameter is greater than any other chord.

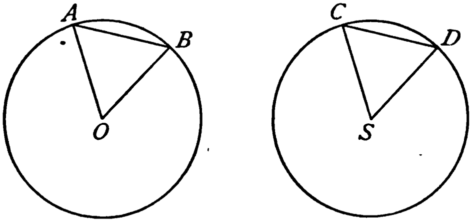
Suggestion. Draw any chord, not a diameter, and draw radii to the extremities. Compare the chord with the sum of the radii.

Ex. 105. If the diagonals of a parallelogram are equal, prove that the parallelogram is a rectangle.

Ex. 106. Given two parallel lines and a transversal. If each of the two interior angles on the same side of the transversal is bisected, prove that the bisectors are perpendicular to each other.

PROPOSITION VII.

181. **Theorem.** CONVERSE OF PROPOSITION V. *In the same circle, or in equal circles, arcs which are subtended by equal chords are equal.*



Let O and S represent equal circles in which the chord A B equals the chord C D.

To prove that the arc A B is equal to the arc C D.

Suggestion 1. Draw radii *O A*, *O B*, *S C*, and *S D*.

2. Compare △s *A O B* and *C S D*. Give auth.

3. Compare ∠s *O* and *S*. Give auth.

4. Complete the demonstration.

Therefore —

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182. A **diameter** of a quadrilateral is a line which joins the middle points of two opposite sides.

Ex. 107. The diameter of a parallelogram divides it into two equal parallelograms.

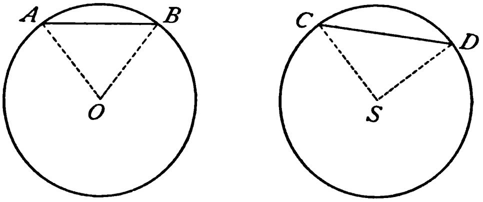
Ex. 108. The two diameters of a parallelogram bisect each other.

Ex. 109. If two adjacent sides of a rectangle are equal the figure is a square.

Ex. 110. Two lines perpendicular, respectively, to each of two intersecting lines, cannot be parallel.

PROPOSITION VIII.

183. Theorem. CONVERSE OF PROPOSITION VI. *In the same circle, or in equal circles, two arcs which are subtended by unequal chords are unequal, and that arc is the greater which is subtended by the greater chord.*



Let O and S represent equal circles, and let the chord AB be less than the chord CD.

To prove that the arc A B is less than the arc C D.

Suggestion 1. Connect the extremities of the chords with the respective centers. Compare ∠ O with ∠ S in △s AOB and CSD. § 139.

2. Compare arc AB with arc CD. Give auth.

Therefore —

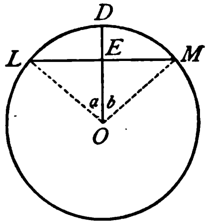
Ex. 111. If a circumference is divided into four equal arcs and chords of the arcs are drawn, the resulting quadrilateral is a square.

Ex. 112. If one of the equal sides of an isosceles triangle is extended through the vertex, and the exterior angle formed is bisected, the bisector is parallel to the base.

Ex. 113. In § 153 draw the diagonals CB and $C'B'$, and prove the proposition by comparison of triangles.

PROPOSITION IX.

184. **Theorem.** *A radius which is perpendicular to a chord of a circle, bisects the chord and its subtended arc.*



Let O D represent a radius perpendicular to the chord L M.

To prove that O D bisects the chord L M and arc L M.

Suggestion 1. Let *E* be the point of intersection of *O D* and *L M*.

2. Compare △s *L E O* and *M E O*.

Complete the demonstration. § 177.

Therefore —

PROPOSITION X.

185. **Theorem.** *A radius which bisects a chord of a circle is perpendicular to the chord.*

Let the radius O D, in figure § 184, bisect the chord L M.

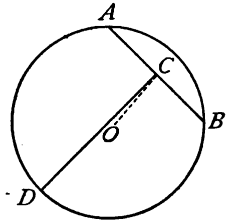
To prove O D is perpendicular to L M.

QUERY: How could this proposition be used to bisect an arc?

Ex. 114. A line which bisects a chord and its subtended arc cuts the center of the circle.

PROPOSITION XI.

186. Theorem. *A line perpendicular to a chord at its middle point, passes through the center of the circle.*



Let A B represent a chord, and C D a perpendicular to A B at its middle point C.

To prove that C D passes through the center of the circle.

Suggestion 1. From *O*, the center of the circle, drop a ⊥ to the chord *A B*. § 184.

2. Where does *C O* lie with respect to *C D?* Why?

ANOTHER METHOD. *Suggestion 1.* What is the locus of a point equally distant from *A* and *B ?*

2. Locate the center *O* with respect to this locus. Give auth.

Therefore —

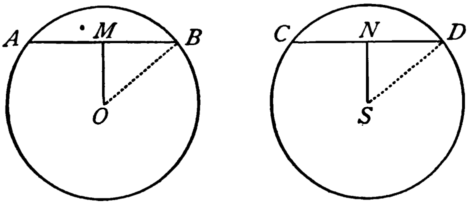
187. COROLLARY.—*A line perpendicular to a chord at its middle point, bisects the subtended arc.*

Ex. 115. If the line joining the middle points of two chords passes through the center of the circle, the chords are parallel.

Ex. 116. A radius drawn to the middle point of an arc bisects its subtending chord.

PROPOSITION XII.

188. Theorem. *In the same circles or in equal circles, equal chords are equally distant from the center, and of two unequal chords, the greater is nearer the center.*



Let A B and C D represent equal chords in the equal circles O and S, and O M and S N their respective distances from the centers O and S.

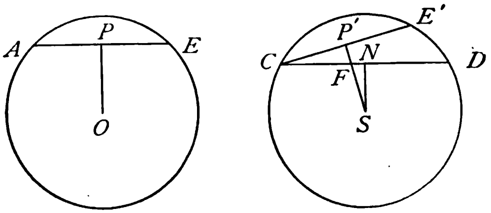
CASE I. *To prove that O M is equal to S N.*

Suggestion 1. Draw the radii *O B* and *S D*.

What relation must *M O* and *N S* bear to *A B* and *C D*, respectively? § 130.

2. Compare △ *O M B* with △ *S N D*.

3. Compare *O M* and *S N*.



Let chord A E be less than chord C D, and let O P and S N represent the distances of the chords from their centers.

CASE II. *To prove that $O P$ is greater than $N S$.*

Suggestion 1. Compare arc $A E$ and $C D$.

2. Place $\odot$ O upon $\odot$ S, the center O upon the center S, and point A upon point C.

Where does point E fall with respect to point D? Sug. 1.

Where does line $A E$ fall with respect to segment $C E' D$?

3. Compare $N S$ and $F S$; $N S$ and $P' S$; $N S$ and $O P$.

Therefore —

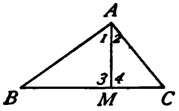
QUERY: Why does $P' S$ cross the line $C D$? Sug. 2.

COROLLARY.—*In the same circle or in equal circles, two chords equally distant from the center are equal; and of two chords unequally distant from the center, that is the greater which is nearer the center.*

Ex. 117. Through a given point within a circle, the smallest possible chord is the one that is perpendicular to the radius passing through the point.

Ex. 118. $A B C$ is an isosceles triangle whose base is $B C$ and whose vertex is A. Extend $B A$ through A to O, making $A O$ equal to $B A$. Connect O and C. Prove that $O C$ is perpendicular to $B C$. (Ex. 112.)

Ex. 119. If a perpendicular is drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, prove that the two triangles formed are mutually equiangular.

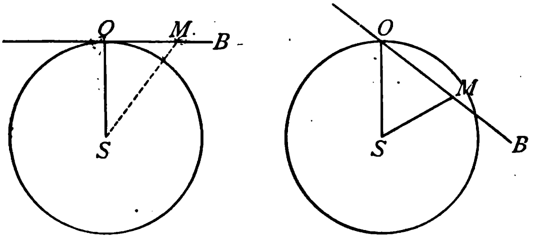


Ex. 120. Prove in Ex. 119 that each segment of the triangle is mutually equiangular to the whole triangle.

Ex. 121. What is the locus of a point equally distant from two parallel straight lines.

Proposition XIII.

189. Theorem. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*



Let O B represent a straight line perpendicular to the radius O S at its extremity O.

To prove that O B is tangent to the circle.

Suggestion 1. What must be proved to know that *O B* is a tangent? § 166.

2 How much of what must be known is given in the theorem?

3. What remains to be proved?

4. Let *M* represent any point except *O* in line *O B*. Draw *S M*.

5. Compare *O S* and *S M* in respect to length.

6. Where is point *M* in respect to the circle?

Therefore —

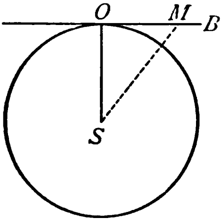
Query: How could this proposition be used to draw a tangent to a circle?

Ex. 122. The bisectors of the interior angles of a non-equilateral parallelogram form a rectangle. Suppose the parallelogram is equilateral?

PROPOSITION XIV.

190. Theorem. CONVERSE OF PROPOSITION XIII.

If a straight line is tangent to a circle, the radius meeting it at the point of tangency is perpendicular to it.



Let O B represent a tangent to the circle S, O the point of tangency, and S O the radius drawn to the point of tangency.

To prove that S O is perpendicular to O B.

Suggestion 1. Draw *S M*, a line from *S* to any point except *O* in *O B*.

2. Compare *S O* and *S M* in respect to length.

3. § 130.

Therefore —

191. COROLLARY 1.—*At any point in a circumference, one, and only one, tangent can be drawn.*

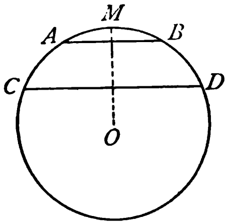
192. COROLLARY 2.—*A straight line perpendicular to a tangent at the point of tangency cuts the center of the circle.*

Ex. 123. All chords of a circle that are tangent to a concentric circle are equal.

Ex. 124. If two tangents are drawn to a circle at the ends of a diameter, they are parallel.

Proposition XV.

193. Theorem. *Arcs of a circle intercepted by parallel lines are equal.*



Let A B and C D represent two parallel lines intercepting the arcs A C and B D.

To prove that arc A C is equal to arc B D.

Suggestion 1. Drop a ⊥ from O to $C D$, and extend it to meet the circumference, as at M.

2. How is $O M$ related to $A B$?

3. Complete the demonstration. § 184.

Therefore —

Review.

194. State all the possible cases in which two parallel lines may intercept two arcs, and prove them equal in each case.

Ex. 125. Two tangents drawn to a circle from the same point are equal. *Suggestion.* Connect the center with the given point. § 190.

Ex. 126. If a tangent and a chord are parallel, prove that they intercept equal arcs.

PROPOSITION XVI.

195. Theorem. *Through three points, not in the same straight line, one circumference, and only one, is possible.*

Let A B C represent three given points not in the same straight line.

To prove that through A, B, and C one circumference, and but one, is possible.

Suggestion 1. What is the locus of a point equally distant from *A* and *C*? Why is this line a locus of the centers of circles whose circumferences cut *A* and *C*?

2. What is the locus of a point equally distant from *A* and *B*? Give auth. Of what centers is this line a locus?

3. Will these two loci intersect? Why? Ex. 110.

4. What relation does this point bear to *A*, *B*, and *C*? Why?

5. Can a circumference pass through the points *A*, *B*, and *C*? Why?

6. Can there be more than one such circumference? Why? § 136.

Therefore —

QUERY: How could you find the center of a circle if it were not known?

MEASUREMENT.

196. **Quantity** is any magnitude which can be measured by a unit of the same kind.

197. The quantities used in geometry are the geometric magnitudes, as, lines, surfaces, and solids.

198. To **measure a quantity** is to find out how many times it contains another selected quantity of the same kind, called the unit of measure.

199. In everyday experience, the **unit of measure** is a standard accepted by general consent; as a foot, a square yard, a ton, a cord, etc.

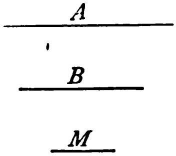
200. The **numerical measure of a quantity** is the number which expresses how many times the unit of measure is contained in the given quantity.

201. A careful distinction should be made between quantity, and number which is the measure of quantity. These terms are sometimes carelessly confused. 26 gallons, 3 feet, 5 pints, and 29 square feet are quantities; 26, 3, 5, and 29 are numbers. The units of measure by which we derive these numbers from the quantities are, respectively, a gallon, a foot, a pint, and a square foot.

202. The **ratio of one quantity to another** is the number of times the first contains the second.

(*a*) In other words the ratio of one quantity to another is the numerical measure of the first regarding the second as the unit of measure; or, having measured both quantities by the same unit, the quotient of the numerical measure of the first divided by the numerical measure of the second.

To illustrate, the ratio of line *A* to line *B* is the number of times *A* contains *B*, regarding *B* as the unit of measure. The result may be obtained by laying *B* off upon *A* as many times as possible; or, *A* and *B* may be measured by the same unit of measure, *M*, in which case the ratio of *A* to *B* is the number of times the numerical measure of *A* contains the numerical measure of *B*. Suppose *M* is contained *c* times in *A*, and *d* times in *B*, then the ratio of *A* to *B* is equal to *c* divided by *d*, as *c* and *d* are the numerical measures of *A* and *B*, respectively.



(1) As ratio plays an important part in subsequent geometry, it is important that the pupil have clear and definite ideas of what a ratio actually is, and the relation it sustains to the subject of division.

From the definition of the ratio of two quantities, as the number of times the first contains the second, it follows that a ratio can exist only between quantities of the same kind, and also that the ratio of two quantities is always a pure number. For example, 6 feet contains 3 feet twice; hence the ratio of 6 feet to 3 feet is 2.

Division has been defined as the process of finding how many times one number or quantity is contained in another number or quantity of the same kind; or, the process of separating a number or quantity into a given number of equal parts and taking one of them. In the first conception of division in the definition, the division of measurement, the quotient is identical with the ratio; the divisor and dividend, if quantities, must have a like unit of measure, making the quotient a pure number, or the division is unthinkable. 6 feet ÷ 3 feet = 2; or, $\frac{6\text{ ft.}}{3\text{ ft.}} = \frac{6}{3} = 2$. This idea may be expressed: 6 feet = 2 times 3 feet.

(2) In the second kind of division, the division of separation or partition, the divisor is a pure number and the dividend and quotient may be numbers, or like quantities. 16 gallons ÷ 2 means that 16 gallons are to be divided into two equal parts, and one of the parts taken. In common phrase, it means to find $\frac{1}{2}$ of 16 gallons. As ratio identifies itself only with the division of measurement, we shall be concerned with the principles underlying only that kind of division.

(3) Since ratio in the following discussion is based upon the alge-

braic conception of division, and not upon the Euclidean definition, which is much too difficult for beginners, the division or fractional form of statement is most expressive. Hence, the ratio of 6 feet to 2 feet is best expressed as $\frac{6 \text{ ft.}}{2 \text{ ft.}}$. The statement referred to, § 202 (a), that the ratio of quantities A and B equals the ratio of the numbers c and d, is best expressed as $\frac{A}{B} = \frac{c}{d}$. This form will be invariably used in the following pages.

203. Two quantities of the same kind are **commensurable** when each contains the same unit of measure an integral number of times.

(*a*) Arithmetic is particularly concerned with commensurable quantities.

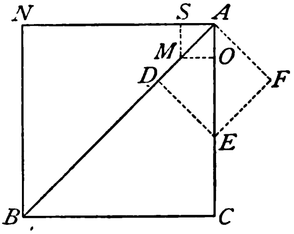
204. In determining the common unit of measure of two commensurable quantities, the arithmetical method for finding the greatest common divisor may be followed, which is, to divide the greater quantity by the less, to divide the divisor by the remainder, this remainder by the second remainder, etc., until the division is found to be exact. The exact divisor is the greatest common unit of measure of the two quantities.

To find the greatest common unit of two lines, or other magnitudes, by the arithmetical method, is a valuable geometrical exercise and useful review of arithmetic.

205. Two quantities of the same kind are **incommensurable** when they cannot be exactly divided by the same unit of measure, however small, an integral number of times.

(*a*) This idea is difficult of conception. To illustrate: The diagonal and side of a square may be shown geometrically to be incommensurable by the following diagram.

By the method of finding the greatest common unit of measure of two quantities, lay off the side $B\,C$ upon the diagonal BA. Erect a $\perp$ to BA at D. By Exercise 80, $AD = DE = EC$. AE is the diagonal of a square of which AD is the side. Lay off AD upon BC, the last divisor, or its equal AC. It is contained twice with a remainder, as AO. Erect a $\perp$ to AC at O. $AO = OM = MD$. AM is the diagonal of a square of which AO is a side. AO is contained twice in AD with a remainder. By continuing the division in the same way, it can be seen that the last remainder is always contained twice in the last preceding divisor with a remainder, for each time the remainder is the difference between the side and the diagonal of a square. The conditions after each division are exactly the same as after the previous division; hence, like conditions must recur without limit; hence, the division can never be exact; hence, there is no common divisor between the side and diagonal of a square; hence, they are incommensurable.



(*b*) The diagonal and side of a square will later be proved to be in the ratio of $\sqrt{2}$ to 1, or $\frac{\sqrt{2}}{1}$, which equals $\sqrt{2}$. The $\sqrt{2}$ can be shown algebraically to be a surd; that is, incommensurable with the unit one.

206. The circumference of a circle and its diameter are incommensurable. The two lines which bear the relation of 1 and $\sqrt{3}$, or 1 and $\sqrt{5}$, are incommensurable. Many quantities in common experience are incommensurable, but as the remainder is small, it is ignored and the quantities are regarded as actually commensurable. In measuring distances the carpenter discards the remainder to the degree of accuracy that his work requires.

Show further illustrations of incommensurables in common experience.

207. An incommensurable number is a number which is incommensurable with its unit, as the $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{11}$, or 3.1416+ etc.

208. An incommensurable ratio is the ratio of any two incommensurable quantities or numbers. The side and diagonal of a square, § 205 (*a*), also the diameter and circumference of a circle, when compared, form an incommensurable ratio.

NOTE.—An incommensurable number must not be looked upon as an inexact number. If the side of a square is 1 foot, the diagonal is $\sqrt{2}$ feet, but this diagonal is a perfectly definite length, and can be constructed as exactly as any other line.

209. A constant is a quantity whose value is fixed.

210. A variable is a quantity which, under the conditions imposed upon it, may assume an indefinite number of values. For example, the distance from a railway station to a moving train is a quantity which has one value at one time, another value a minute later, and still different values later on.

211. When a variable so changes that it continually approaches a constant which it cannot reach, but from which it may be made to differ by an amount less than any assigned quantity, however small, the constant is the **limit of the variable**, and the variable approaches its limit.

NOTE.—By this definition some variables have limits and some do not. The variable in the illustration § 210 belongs to the latter class; the variables considered in geometry belong to the former.

(*a*) Tests for the limit of a variable:

1. Is it a constant?
2. Is it approached in value by a variable?
3. Can it be equaled by the variable?
4. Can the difference between the variable and its limit be made less than any assigned quantity?

(*b*) If a point is made to move along a given line, the distance from a fixed point to the moving point is a variable. This variable may or may not have a limit. If such conditions are imposed upon the moving point that, during a given period of time, it moves along half the length of the line, the next period along half the remaining distance, the next half the remaining distance, etc., the length of the given line is the limit of the variable.

(*c*) One *m*th of a given line is a variable which approaches zero as a limit if *m* is made to increase indefinitely; *a* minus one *m*th of *a* is a variable which approaches *a* as a limit if *m* is made to increase indefinitely.

Illustrate freely the idea of variable in the article above; as, if 16 is divided by an increasing series of numbers, as 2, 4, 8, 16, 32, 64, 128, etc., the quotient is a variable approaching zero as its limit.

(*d*) The perimeter of a polygon consisting of equal chords of a circle is a variable, if the number of its sides is continually increased by drawing chords from the vertices of the polygon to the middle points of the arcs.

What is the limit of this perimeter?

Test your answer by the definition of limit.

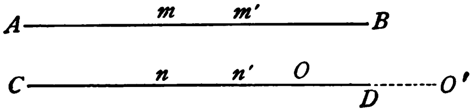
(*e*) The area of the polygon is also a variable under the above conditions. What is its limit?

(*f*) The pupil should remember that the limit and the variable are always the same kind of quantities; as in example (b), the limit and variable are both distances; in (d), are both lines, etc.

212. The difference between a variable and its limit is a variable whose limit is zero.

Proposition XVII.

213. Theorem. *If two variables are always equal as they approach their limits, their limits are equal.*



Let Am and Cn represent two equal variables, and AB and CD their respective limits.

To prove that AB equals CD.

AB and CD are either equal or unequal. Suppose AB is smaller than CD. Let $AB = CO$. Lay off AB on CD; then CO becomes the limit of the variable Am.

By hypothesis $Am = Cn$, $Am' = Cn'$, etc. By definition of limit of a variable, Am approaches indefinitely near CO and Cn approaches CD. Hence, as OD is an assigned quantity to denote the difference between AB and CD, the variable Am eventually becomes smaller than Cn as they together approach their limits. This is contrary to the hypothesis, hence AB cannot be smaller than CD. In a similar manner prove AB cannot be larger than CD.

Therefore—

The absurdity of the assumption of the inequality of

AB and CD can be shown also in a violation of the definition of limit of a variable.

In the above demonstration, if the variables maintain their equality, one of them cannot approach within less than an assigned distance of its limit, or the other must become larger than its limit. If Am remains equal to Cn, as Cn approaches its limit CD, it must become larger than CO its limit; which violates the definition of limit of a variable.

214. POSTULATES. (*a.*) *The variable which is obtained by multiplying a given variable by a constant has for its limit the product of the limit of the variable by the constant.*

If V is a variable, L its limit, and A any constant, $A \times V$ approaches $A \times L$ as its limit.

The following may serve as a numerical illustration:

$$2 \times (.3, .33, .333 \ldots . \tfrac{1}{3}) = .6, .66, .666 \ldots . \tfrac{2}{3}.$$

(*b.*) *The variable which is obtained by dividing a given variable by a constant, has for its limit the quotient of the limit of the given variable by the constant.*

If V is a variable, L its limit, and A any constant, $\frac{V}{A}$ approaches $\frac{L}{A}$ as its limit.

To illustrate: $(.6, .66, .666 \ldots . \tfrac{2}{3}) \div 2 = .3, .33, .333 \ldots . \tfrac{1}{3}.$

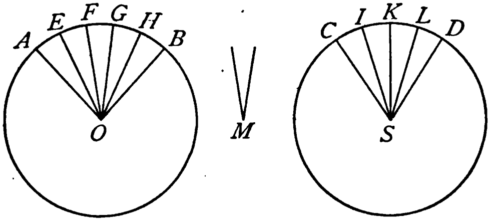
(*c.*) *If two variables, both increasing or both decreasing, are multiplied together term for term the product is a variable, the limit of which is the product of the limits of the variables.*

If V is one variable, and V' another, L and L' their respective limits, both variables increasing or both decreasing, $V \times V'$ is a variable, and $L \times L'$ is its limit.

To illustrate: Multiply the above numerical variables term for term.

PROPOSITION XVIII.

215. Theorem. *In the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept at the circumference.*



Let O and S represent angles at the center of two equal circles, and A B and C D the arcs which they intercept at the circumference.

To prove that $\frac{\text{angle } O}{\text{angle } S}$ *is equal to* $\frac{\text{arc } AB}{\text{arc } CD}$.

There are two cases.

CASE I. *When angles O and S are commensurable.*

Suggestion 1. Let $\angle M$ represent a common unit of measure for the two angles.

2. If $\angle M$ is contained 5 times in $\angle O$ and 4 times in $\angle S$, what does the ratio $\frac{\angle O}{\angle S}$ equal? § 202.

Or, if $\angle M$ is contained a times in $\angle O$ and b times in $\angle S$, what does the ratio of $\frac{\angle O}{\angle S}$ equal?

3. Extend the lines of division of the $\angle$s to the arcs. How do arcs AE, EF, CI, IK, etc., compare? Why? Hence, one of them may be used as a unit of measure.

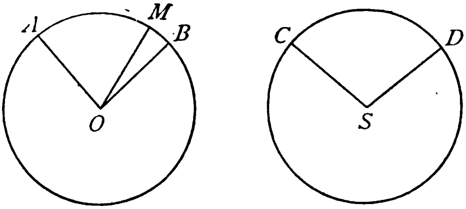
4. How many times is this unit arc contained in arc *A B*? In arc *C D*?

5. Then the ratio $\frac{arc\ A\ B}{arc\ C\ D}$ equals what? § 202.

6. Compare the ratio of the ∠s with the ratio of the arcs. Give auth.

Therefore —

CASE II. *When the angles O and S are incommensurable.*



Suggestion 1. Take any unit of measure of one of the angles, as of ∠ *S*, and apply this unit to the ∠ *O*. There must be a remainder. Why?

Let this remainder be represented by ∠ *M O B*. How large is this remainder with respect to the unit of measure? Why?

2. The ∠s *A O M* and *S* are commensurable. Why?

3. Compare the ratio $\frac{\angle A\ O\ M}{\angle S}$ with the ratio $\frac{arc\ A\ M}{arc\ C\ D}$. Case I.

4. By taking the unit of measure of ∠ *S* smaller and smaller continually, the remainder in ∠ *A O B*, namely, the ∠ *M O B*, may be made as small as we please, that is, it may be made to approach zero at its limit. (§ 212.) Can it be made to disappear entirely? Why? Sug. 1.

5. The ∠ *A O M* is a variable. Why?

6. Is $\angle S$ a variable or a constant?

7. Is the ratio $\frac{\angle AOM}{\angle S}$ a variable or constant? What *ratio* is $\frac{\angle AOM}{\angle S}$ approaching? § 214 (*b*).

9. What kind of a quantity is arc AM? Is arc CD? The ratio of $\frac{arc\ AM}{arc\ CD}$?

10. What is the limit of $\frac{arc\ AM}{arc\ CD}$? § 214 (*b*).

11. Compare the two variables $\frac{\angle AOM}{\angle S}$ and $\frac{arc\ AM}{arc\ CD}$ as they approach their limits. Sug. 3.

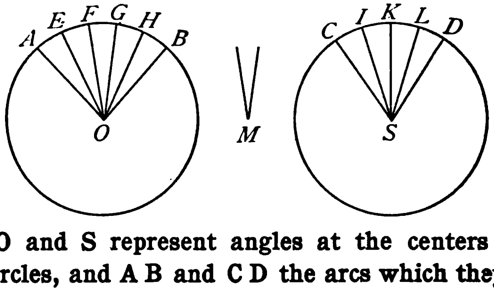
12. Compare their limits. § 213.

Therefore —

Model.

PROPOSITION XVIII.

Theorem. *In the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept at the circumference.*



Let O and S represent angles at the centers of two equal circles, and A B and C D the arcs which they intercept at the circumference.

To prove that $\dfrac{\text{angle } O}{\text{angle } S}$ *is equal to* $\dfrac{\text{arc } A\,B}{\text{arc } C\,D}$.

There are two cases.

CASE I. *When angles O and S are commensurable.*

Let $\angle M$ represent a common unit of measure of $\angle O$ and $\angle S$.

If $\angle M$ is contained 5 times in $\angle O$ and 4 times in $\angle S$, the ratio $\dfrac{\angle O}{\angle S} = \dfrac{5}{4}$. § 202.

Or, if $\angle M$ is contained a times in $\angle O$ and b times in $\angle S$, the ratio $\dfrac{\angle O}{\angle S} = \dfrac{a}{b}$. § 202.

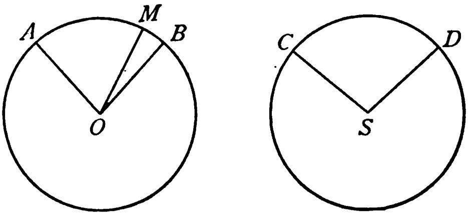
Extend the lines of division of the $\angle$s to the arcs; then arcs $A\,E$, $E\,F$, $C\,I$, $I\,K$, etc., are all equal, and hence the arcs $A\,B$ and $C\,D$ can be measured by using any one of them as a unit of measure. § 177, § 198

The unit arc is contained 5 times in arc $A\,B$ and 4 times in arc $C\,D$, or a and b times respectively, for each angle at the center intercepts an arc.

Hence, the ratio $\dfrac{\text{arc } A\,B}{\text{arc } C\,D} = \dfrac{5}{4}$ or $\dfrac{a}{b}$. § 202.

$\therefore \dfrac{\angle O}{\angle S} = \dfrac{\text{arc } A\,B}{\text{arc } C\,D}$. *Ax.* 1.

CASE II. *When angles O and S are incommensurable.*



Take a unit of measure of one of the $\angle$s, as of $\angle S$, and apply this unit to the $\angle O$. Since the angles are incommensurable, this unit will be contained in $\angle O$ a certain number of times with a remainder, as $\angle M O B$, which is less than the unit of measure. § 205.

Since the $\angle$s $A O M$ and S have a common unit of measure, they are commensurable, and hence, by Case I,

$$\frac{\angle A O M}{\angle S} = \frac{arc\ A M}{arc\ C D}.$$

By taking the unit of measure of $\angle S$ smaller and smaller continually, the remainder in $\angle A O B$, viz., the $\angle M O B$, which is always less than the unit, may be made indefinitely small, but cannot be made entirely to disappear, for then the $\angle$s O and S would be commensurable. Hence, the remainder, $\angle M O B$, approaches zero as a limit, and, therefore, the $\angle A O M$ is a variable which approaches $\angle A O B$ as a limit. § 211.

But $\angle S$ is a constant, and hence, the ratio $\frac{\angle A O M}{\angle S}$ is a variable which approaches the ratio $\frac{\angle A O B}{\angle S}$ as a limit. § 214.

Arc $A M$, as $\angle A O M$ increases, is made to increase (§ 177), and hence is a variable, which approaches arc $A B$ as a limit (§ 211). Arc $C D$ is a constant, hence the ratio $\frac{arc\ A M}{arc\ C D}$ approaches the ratio $\frac{arc\ A B}{arc\ C D}$ as a limit. § 214.

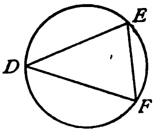
As $\frac{\angle A O M}{\angle S}$ and $\frac{arc\ A M}{arc\ C D}$ are two variables which are always equal as they approach their limits, their limits must be equal (§ 213). That is, $\frac{\angle AOB}{\angle CSD} = \frac{\text{arc } AB}{\text{arc } CD}$.

Therefore, in the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept on the circumference.

216. Scholium I. A degree, which has already been defined as $\frac{1}{90}$ of a right angle, or $\frac{1}{360}$ of the whole angular magnitude about a point, is the standard unit angle; and the arc it intercepts on the circumference, also called a degree, or $\frac{1}{360}$ of the circumference, is the standard unit arc.

217. Scholium II. By the proposition it is proved that the number of times the unit angle is contained in the given angle at the center is the same as the number of times that the unit arc is contained in the arc intercepted by the given angle. Hence, we say briefly that **an angle at the center is measured by its intercepted arc,** meaning that there are as many degrees in the angle at the center as in the intercepted arc.

218. Scholium III. A right angle which contains 90 degrees (written 90°), is often used as a unit angle in determining the size of other angles.

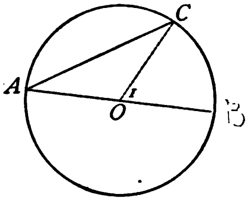


219. An angle **inscribed in a circle** is one whose vertex is in the circumference, and whose sides are chords of the circle. The angle E is an inscribed angle.

220. An angle is **inscribed in a segment** of a circle when its vertex is in the arc of the segment and its sides meet the extremities of the chord of the arc. The angle E is inscribed in the segment DEF.

PROPOSITION XIX.

221. Theorem. *An inscribed angle is measured by one-half its intercepted arc.*



Let A be an inscribed angle, intercepting the arc B C.

To prove that the angle A is measured by one-half the arc B C. § 217.

CASE I. *When one side of the angle is a diameter.*

Suggestion 1. Connect O, the center of the circle, with C.

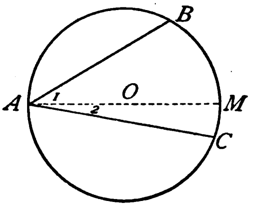
2. Compare $\angle A$ with $\angle C$. Give auth.

3. Compare $\angle A$ with $\angle 1$. Give auth.

4. By what arc is $\angle 1$ measured? Why? § 217.

5. Then, by what part of the arc $B C$ is the $\angle A$ measured?

CASE II. *When the center lies between the sides of the angle.*



Suggestion 1. Through the vertex A draw the diameter $A M$.

2. By what arc is ∠ *1* measured? Why?

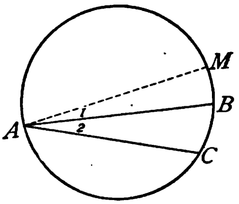
3. By what arc is ∠ *2* measured? Why?

4. Then, by what arc is ∠ *A* measured?

NOTE.—Suggestion 4 may be more easily comprehended if the definition of "measured" (§ 217) be substituted; as, the number of angle degrees in ∠ *1* plus the number of angle degrees in ∠ *2* = $\frac{1}{2}$ the number of arc degrees in *B M* plus $\frac{1}{2}$ the number of arc degrees in *M C*.

Express algebraically and reduce by factoring.

CASE III. *When both sides of the angle are on the same side of the center.*



Suggestion 1. Through the vertex *A* draw the diameter *A M*.

2. By what arc is ∠ *A* measured?

3. By what arc is ∠ *1* measured?

4. Then, by what arc is ∠ *2* measured? (See Note, Case II.)

Therefore—

222. COROLLARY I. *An inscribed right angle is measured by one-half a semi-circumference.*

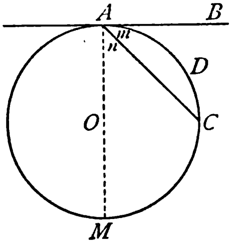
223. COROLLARY II. *An angle inscribed in a semi-circle is a right angle.*

QUERY. How could you use Corollary II to construct a right angle?

EX. 127. If angle *A*, § 221, Case I = 63½°, how many degrees in angle 1? In angle *C*? In arc *A C*?

PROPOSITION XX.

224. Theorem. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*



Let *m* represent an angle formed by A B a tangent, A C a chord, and A D C the intercepted arc.

To prove that the angle m is measured by one-half the arc A D C.

Suggestion 1. Draw through *A* the diameter *A M*. What kind of an ∠ is *B A M* ? Why? By what arc is it measured? § 95.

2. By what arc is ∠ *n* measured? Why?

3. By what arc is ∠ *m* measured? Why? (See Note § 221.)

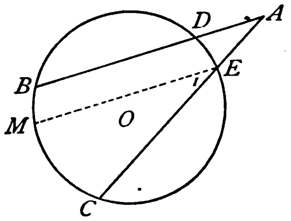
Therefore—

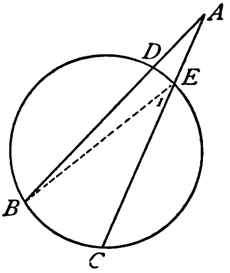
Ex. 128. A chord is met at its extremity by a tangent forming an angle of 75°. How many degrees in the arc that is subtended by the chord?

Ex. 129. *a*, *b* and *c* are the angles of an inscribed triangle. Angle *a* is four times *b*, and *b* is one-seventh of *c*. How many degrees in the arcs of the circle subtended by the sides of the angles, respectively?

PROPOSITION XXI.

225. Theorem. *The angle formed by two secants, meeting without the circle, is measured by one-half the difference of the intercepted arcs.*





Let A B and A C represent two secants, meeting at A, without the circle, forming the angle B A C and intercepting the arcs D E and B C.

To prove that the angle B A C is measured by one-half the difference of the arcs B C and *D E.*

Suggestion 1. Through *E*, draw a chord *E M* $\parallel$ to *A B.*

2. By what arc is $\angle 1$ measured? Why?

3. Compare the $\angle A$ with $\angle 1$.

4. Then, by what arc is $\angle A$ measured?

5. Express the arc *M C* in terms of *B C* and *B M*, then in terms of *B C* and *D E.*

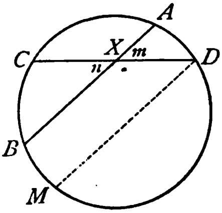
6. Then, by what arc is $\angle A$ measured?

Therefore—

ANOTHER METHOD. Connect *B* and *E*. Compare $\angle A$ with $\angle B$ and $\angle 1$. Complete the demonstration.

PROPOSITION XXII.

226. Theorem. *An angle formed by two intersecting chords is measured by one-half the sum of the intercepted arcs.*



Let A B and C D represent two chords intersecting at X.

To prove that angle n is measured by one-half the sum of arcs C B and A D.

Suggestion 1. Through D, one extremity of the chord $C D$, draw $D M \parallel$ to $A B$.

2. Compare $\angle n$ with $\angle D$. Give auth.

3. By what arc is $\angle D$ measured? Why?

4. By what arc is $\angle n$ measured?

5. Express the arc CM in terms of BC and AD. § 193.

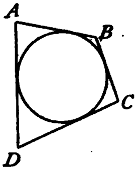
6. Then, by what arc is $\angle n$ measured in terms of BC and $A D$?

Therefore—

ANOTHER METHOD. Connect B and D. Compare $\angle n$ with $\angle B$ and $\angle D$. Complete the demonstration.

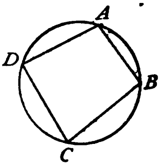
QUERY. What angles have we learned to measure? What is the measure of each?

227. A polygon is **circumscribed about a circle,** when



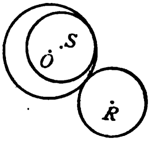
each of its sides is tangent to the circle; as polygon *A B C D*.

(*a*) When a polygon is circumscribed about a circle, the circle is **inscribed in the polygon.**



•**228.** A polygon is **inscribed in a circle** when each of its sides is a chord of the circle; as polygon *A C*.

(*a*) When a polygon is inscribed in a circle, the circle is **circumscribed about the polygon.**



229. Two **circles are tangent** to each other when they have one point of contact, and only one, as circles *O* and *R*, or *O* and *S*.

(*a*) This point of contact is called the **point of tangency.**

Ex. 130. An angle formed by two tangents is measured by one-half of the difference between the intercepted arcs.

Ex. 131. An angle formed by a tangent and a secant is measured by one-half of the difference between the intercepted arcs.

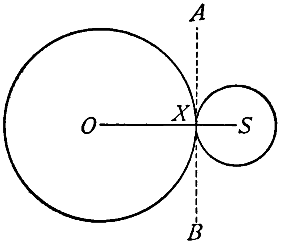
Ex. 132. If the middle points of the sides of a quadrilateral be joined in order, the figure formed is a parallelogram. *Sug.* Draw the diagonals of the quadrilateral.

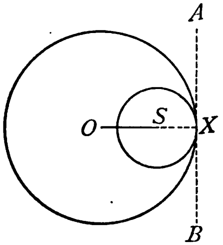
Ex. 133. The diameters of any quadrilateral bisect each other.

Ex. 134. If *A* is the vertex and *B C* the base of an isosceles triangle *A B C*, and if from any point *D* in the side *A B* a line is drawn perpendicular to the base and meeting *C A* extended at *E*, prove that the triangle *A D E* is isosceles.

Proposition XXIII.

230. **Theorem.** *If two circles are tangent to each other, the line joining their centers passes through the point of tangency.*





Let circles O and S be tangent to each other, X the point of tangency and O S the line joining the centers.

To prove that O S, extended if necessary, passes through the point of tangency X.

Suggestion 1. Represent a straight line *A B* that is a tangent common to the two circles.

2. At the point of tangency *X*, erect a ⊥ to *A B*, and extend it through both circles.

3. The ⊥ passes through the centers. Why?

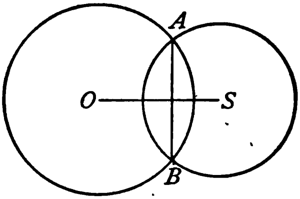
4. What relation does this ⊥ sustain to *O S*? Give auth. *Ax. 12.*

Therefore—

Ex. 135. *A B* is the hypotenuse of a right triangle *A B C*, *B D* is drawn bisecting the angle *B*, meeting *A C* at *D*, and *D E* is drawn perpendicular to *A C*, meeting *A B* at *E*. Prove that *E D B* is an isosceles triangle.

PROPOSITION XXIV.

231. Theorem. *If two circles intersect, the line joining their centers is perpendicular to their common chord at its middle point.*



Let circles O and S intersect. Let O S be the line joining their centers and A B their common chord.

To prove that O S is perpendicular to A B at the middle point of A B.

Suggestion. At the middle point of *A B* erect a ⊥ and extend both ways. (§ 186.) Complete the demonstration.

Therefore—

ANOTHER METHOD:

Suggestion 1. What is the locus of points equally distant from *A* and *B*?

2. What relation do the centers bear to *A* and *B*?

3. What relation must they bear to the locus?

Therefore—

232. COROLLARY I. *If two intersecting circles are equal, the line of centers is bisected by the common chord.*

To prove O S is bisected.

QUERY. How could § 232 be used to bisect a straight line?

233. Review.

I. In the same circle or in equal circles:

(*a*) If two angles at the center are equal, compare the subtending arcs; the subtending chords.

(*b*) If two angles at the center are unequal, compare the subtending arcs; the subtending chords.

(*c*) If two arcs are equal, compare the subtended angles at the center; the subtending chords.

(*d*) If two arcs are unequal, compare the subtended angles at the center; the subtending chords.

(*e*) If two chords are equal, compare the arcs they subtend; the angles at the center they subtend.

(*f*) If two chords are unequal, compare the arcs they subtend; the angles at the center they subtend.

II. What lines cut the centers of circles?

III. What angles have been measured? What arcs are the respective measures of the angles?

Problems of Construction.

234. In the previous work the constructions have been represented, instead of being actually performed. If it has been established that a certain relation of points, lines or surfaces is possible, the rigor of the demonstration is not impaired by using a representation of that relation without actually constructing it.

235. Since plane geometry deals only with figures which can be made from straight lines and circumferences of circles, the constructions of plane geometry are those which it is possible to effect by means of a straight edge and dividers, or compasses, which are necessitated by the postulates named below.

Problems of construction belong in no sense to pure geometry, but are applications of the principles demonstrated in pure geometry.

236. Since constructions are based upon theorems previously demonstrated, if several theorems bearing upon the same point have been established, more than one method of solution of the problem involving that point can usually be effected; as, if the problem be given to draw through a given point a line parallel to a given line, several methods for the solution of the problem can be evolved by means of several of the propositions concerning parallel lines, as §§ 114–116.

Postulates of Construction.

237. Let it be granted:

(*a*) That a straight line can be drawn between any two points and can be extended to any length through either extremity.

(*b*) That a circle can be drawn with any point as a center, and with any straight line as a radius.

NOTE.—It will be observed that the first postulate necessitates the straight edge, and the second the dividers.

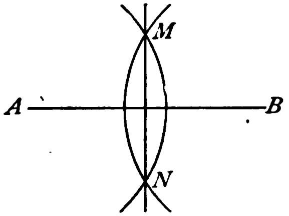
Ex. 136. If the middle points of the three sides of a triangle be joined by straight lines, the triangle is divided into four triangles, which are equal in all respects.

Ex. 137. Two circles are tangent internally. Two lines are drawn from the point of tangency through the extremities of a diameter of one of the circles. Prove that they intersect the other circle in the extremities of a diameter.

Ex. 138. Prove Ex. 137 is true if the circles are tangent externally.

PROPOSITION XXV.

238. Problem. *To bisect a given straight line.*



Let A B be given a straight line.

To bisect A B.

Suggestion 1. Use § 232.

2. To use this truth, *A* and *B* must be the centers of two equal intersecting circles. Hence, with these points as centers, construct two equal intersecting circles (§ 237 (*b*)), and draw their common chord *M N* (§ 237 (*a*)).

Therefore—

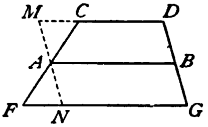
What postulates are used?

Try § 231 for another solution.

QUERY. At what angle does *M N* bisect *A B*?

NOTE.—A little experience will suggest how to omit the unessential parts of lines. For instance, in the above construction all the circumferences can be omitted, except short arcs near the points of intersection, *M* and *N*.

239. The **median** of a trapezoid is the straight line joining the middle points of the legs.

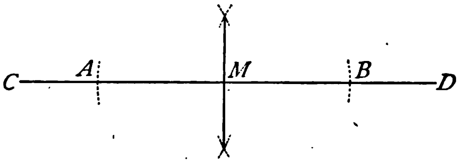


Ex. 139. The sum of the bases of a trapezoid is equal to twice the median of the trapezoid.

Ex. 140. The median of a trapezoid is parallel to the bases.

PROPOSITION XXVI.

240. Problem. *To erect a perpendicular to a given line, at a given point in that line.*



Let C D be the given line and M the given point.

To erect a perpendicular to C D at M.

Suggestion 1. Use § 232.

2. In order to use this principle, *M* must be the middle point of the line joining the centers of the ⊙s. Hence, find points on the line *C D*, as *A* and *B*, equally distant from *M* (§ 237 (*b*)), and proceed as in § 238.

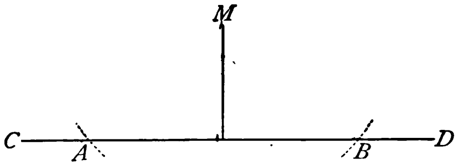
3. A perpendicular is erected to *C D* at *M*.

Therefore—

For other methods, see §§ 185, 231, Ex. 29.

PROPOSITION XXVII.

241. Problem. *From a given point, to drop a perpendicular to a given line.*



Let C D represent the given line, and M the given point.

To drop a perpendicular from M to the line C D.

Suggestion 1. As *M* is one point in the required ⊥, but one other point has to be determined. *Ax.* 12, Cor. 2.

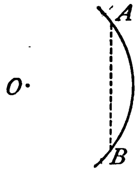
2. Effect a construction so that you can use the truth in § 185.

Therefore—

QUERY. What postulates and problems are used in this problem? §§ 237 (*b*), 238 and 237 (*a*).

PROPOSITION XXVIII.

242. Problem. *To bisect a given arc.*



Let A B represent the given arc.

To bisect the arc A B.

Suggestion 1. What propositions have been demonstrated which involve the bisection of an arc? §§ 184, 185, 187.

2. If the center, as *O*, is given, apply § 184 or § 185 and complete the solution.

3. If the center is not given, apply § 187.

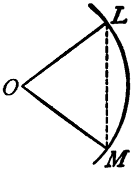
Therefore—

Ex. 141. Let *a* be a given straight line, and *b* an indefinite straight line intersecting it. Construct a right triangle having *a* for its hypotenuse and having its vertex in *b*. § 223.

Ex. 142. *A B* and *C D* are two chords of a circle intersecting at *O*. Prove triangles *A O D* and *C O B* are mutually equiangular. Prove triangles *A O C* and *B O D* mutually equiangular.

PROPOSITION XXIX.

243. Problem. *To bisect a given angle.*



Let O represent a given angle.

To bisect the angle O.

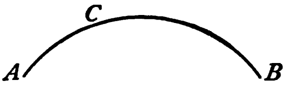
Suggestion 1. What propositions involve the bisection of an ∠? Make the construction by one or more methods.

§§ 242, 178, Ex. 54, etc.

Therefore—

PROPOSITION XXX.

244. Problem. *To find the center of a circle, if any arc of the circumference is given.*



Let A B represent an arc.

To find the center of a circle of which A B is an arc.

Suggestion 1. How many lines must pass through a given point to determine it?

2. What principles have been demonstrated which determine lines that pass through the center of a ⊙?

§§ 186, 192, Ex. 114, etc.

3. Use § 186 to determine two loci of the center.

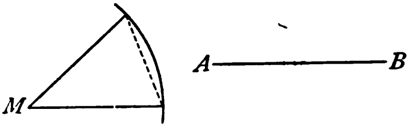
4. Why is their point of intersection the required center? § 195.

Therefore—

NOTE.—Only those truths previously proved that involve postulates, or problems that have been solved, can be used in the solution of a problem. For example, § 186 leads to a possible solution of the problem, § 244, because the problem to erect a perpendicular has already been solved. Can § 192 be used? Why? What is the objection to the use of Ex. 114?

Proposition XXXI.

245. Problem. *At a given point in a given line, to construct an angle equal to a given angle, with the given line as one side.*



Let M represent the given angle, A B the given line, and A the given point.

To construct at A an angle equal to the angle M, having A B for one side.

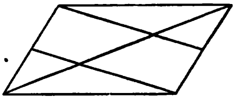
Suggestion 1. Use §§ 173, 181, and 178.

2. To apply these truths, M and A must be made centers of = ⊙s. How can this be done?

3. Equal arcs must be made to subtend ∠ M and the ∠ to be constructed at A. How can this be done? Complete the construction.

Therefore—

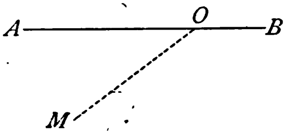
Ex. 143. If from two opposite vertices of a parallelogram two lines be drawn to the middle points of two opposite sides, the lines will trisect the diagonal joining the other vertices. (See Ex. 101.)



Ex. 144. $A B C$ is a triangle inscribed in a circle whose center is O. $O D$ is perpendicular to $B C$. Prove angle $D O C$, or its supplement, is equal to angle A. *Sug.*: O may be within or without the triangle.

PROPOSITION XXXII.

246. Problem. *Through a given point, to construct a straight line parallel to a given straight line.*



Let A B be a given straight line, and M a given point.

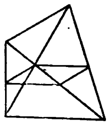
To construct a straight line through M, parallel to A B.

Suggestion 1. What propositions determine that lines are ‖ ? §§ 102, 114, 115, etc.

2. These truths require a transversal through *M*, cutting *A B*. Use § 114. Use § 102. Complete the constructions.

OTHER METHODS. If time permits, try to use other truths by which a line ‖ to another line is determined; as, for instance, §§ 152 and 140 (*c*).

Ex. 145. If the middle points of two opposite sides of a quadrilateral be joined to the middle points of the diagonals, the joining lines form a parallelogram.



Ex. 146. Prove that the three bisectors of the angles of a triangle meet in the same point.

Suggestion 1. In triangle *A B C* the bisector of ∠ *A* is the locus of a point equally distant from the sides *A B* and *A C*. Ex. 71.

2. The bisector of ∠ *B* is the locus of what?

3. Prove, now, that the point of intersection of these two bisectors is on the bisector of the ∠ *C*.

PROPOSITION XXXIII.

247. Problem. *To construct (1) the complement of a given angle; (2) the supplement of a given angle; (3) the third angle of a triangle, having given the other two.*

Suggestion 1. See definitions of complement and supplement of an angle.

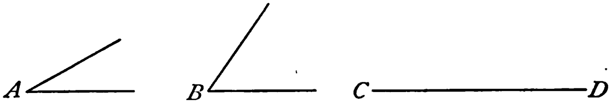
2. In solving (1), § 240.

3. In solving (3), § 51 or § 120.

Therefore—

PROPOSITION XXXIV.

248. Problem. *Given two angles and the included side of a triangle, to construct the triangle.*



Let A and B represent the two given angles, and C D the included side.

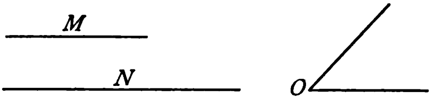
To construct a triangle having A and B for two of its angles, and C D the side included between these angles.

Suggestion. Represent the △ as if already constructed. Study it to see what previous problems need to be employed in the construction.

QUERY. Could the line and angles be of such magnitudes that the construction of a triangle is impossible?

PROPOSITION XXXV.

249. Problem. *Given two sides and the included angle of a triangle, to construct the triangle.*

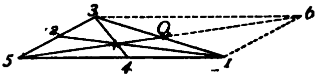


Let M and N represent two sides and O the included angle of a triangle.

To construct a triangle having M and N for two of its sides, and O the angle formed by those two sides.

Suggestion. See Sug. § 248.

Ex. 147. The three medians of a triangle meet at a point. (See Ex. 143.) $1\,2$ cuts off one-third of diagonal $5\,6$. Find relation of $3\,4$ to diagonal. Show that median $5\,O$ lies in diagonal $5\,6$.



Ex. 148. Prove that the line which bisects an arc and is perpendicular to its subtending chord passes, if extended, through the center of the circle.

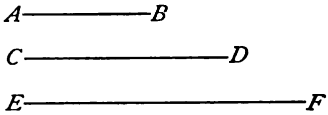
Try this example in the construction of problem, § 244.

Ex. 149. Parallel straight lines, included between parallel straight lines, are equal.

Ex. 150. If a line be drawn in a trapezoid, bisecting one of the legs and parallel to the bases, prove that it bisects the other leg also. *Sug.:* Draw a diagonal of the trapezoid.

PROPOSITION XXXVI.

250. Problem. *To construct a triangle whose sides are three given lines.*



Let A B, C D, and E F represent three given lines.

To construct a triangle whose sides are A B, C D, and E F, respectively.

Suggestion 1. Draw one line as *A B*. Two vertices of the △ are determined. The problem now is to determine the third vertex, *X*.

2. Having constructed line *A B*, what is one locus of the vertex? § 158 or Ex. 100.

3. What is another locus of the vertex?

4. Do these loci intersect?

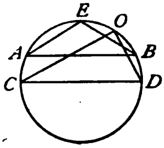
5. Having located point *X*, construct the △.

QUERY. Could *C D* and *E F* be of such length with respect to *A B* that the loci could not intersect? Discuss the possibility of the solution as dependent upon the relative lengths of the three given lines. § 87.

Ex. 151. Upon a given base, construct an isosceles triangle, in which the sum of the two equal sides equals a given line.

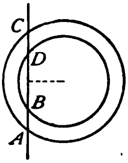
Ex. 152. All angles inscribed in the same segment are equal.

Ex. 153. In the same circle or in equal circles, an angle inscribed in the smaller of two segments is larger than an angle inscribed in the larger segment.



Suggestion. Prove that the angle *A E B* is larger than the angle *C O D*.

Ex. 154. The segments of a straight line intercepted by concentric circles are equal.

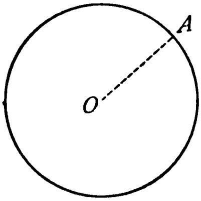


Prove *A B* equals *C D*.

PROPOSITION XXXVII.

251. Problem. *Through a given point, to draw a tangent to a given circle.*

CASE I. *When the point is on the circumference.*



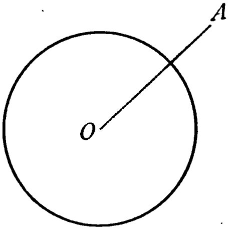
Let A represent the given point in the circumference of the given circle, O.

To draw a tangent to the circle O, through the point A.

Suggestion 1. What relation does a tangent bear to the radius drawn to the point of contact? Give auth.

What problems and postulates are involved in making application of the authority you use?

CASE II. *When the given point is without the circle.*



Let A represent the given point without the circle O.

To draw a tangent to the circle O, through the point A.

Suggestion 1. The problem is to determine the point of tangency. Draw line *O A*.

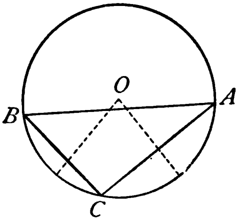
2. If the required point were connected with both *O* and *A*, what kind of an ∠ would be formed at that point? Give auth.

3. § 189.

QUERY. How many tangents can be constructed from a point to a circle?

PROPOSITION XXXVIII.

252. Problem. *To circumscribe a circle about a given triangle.*



Let A B C represent the given triangle.

To circumscribe a circle about the triangle A B C.

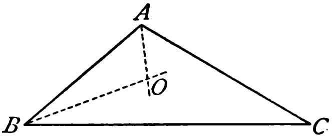
Suggestion. The problem is to find the center of a ⊙ whose circumference passes through *A*, *B*, and *C*; *i. e.*, to find a point equally distant from *A*, *B*, and *C*.

§ 137 or 195.

QUERY. How many circles can be circumscribed about a triangle? Why?

PROPOSITION XXXIX.

253. Problem. *To inscribe a circle in a given triangle.*



Let A B C represent the triangle.

To inscribe a circle in the triangle A B C.

Suggestion 1. Represent a ⊙ inscribed in a △. Study it.

§ 227.

2. What is the locus of a point equally distant from *A B* and *A C*? From *A B* and *B C*? §§ 131, 190, and Ex. 71.

3. Give complete directions for inscribing a ⊙ in a △.

QUERY. How many circles can be inscribed in a given △? Why?

Ex. 155. If tangents to a circle be drawn at the extremities of any chord, these tangents make, with each other, an angle which is twice the angle between the chord and the diameter of the circle drawn through the extremity of the chord.

CHAPTER III.

PROPORTIONAL LINES, AND SIMILAR POLYGONS.

The Theory of Proportion.

254. **A proportion** is an equality between ratios.

(*a*) Let $\frac{A}{B}$ represent a ratio and $\frac{C}{D}$ an equal ratio, then $\frac{A}{B} = \frac{C}{D}$ is a proportion.

(*b*) It may be written in the above form; as $A : B :: C : D$, or, as $A : B = C : D$, and is read: the ratio of A to B equals the ratio of C to D; or, A is to B as C is to D.

The reading, A is to B as C is to D, belongs especially to the form, $A : B :: C : D$.

255. The **terms** of a proportion are the four numbers, or quantities, compared.

256. In any ratio the first term is the **antecedent,** and the second term the **consequent.** Hence, in any proportion the first and third terms are antecedents and the second and fourth terms are consequents.

257. In any proportion, the first and fourth terms are the **extremes,** and the second and third terms are the **means.**

In the proportion $\frac{A}{B} = \frac{C}{D}$, the first term is A, the second B, the third C, and the fourth D. A and C are antecedents, B and D are consequents, A and D are the extremes, and B and C are the means.

258. The terms of a ratio must be numbers, or quantities of the same kind. §§ 198 and 202.

259. A **fourth proportional** to three numbers or quantities is the fourth term of a proportion in which the three terms are the three quantities taken in order. X is a fourth proportional to A, B, and C, if $\frac{A}{B} = \frac{C}{X}$. 12 is a fourth proportional to 6, 9, and 8 if $\frac{6}{9} = \frac{8}{X}$ Verify the last statement.

PROPOSITION I.

260. Theorem. *In a proportion, the product of the means equals the product of the extremes.*

CASE I. *If all the terms are numbers.*

Let $\frac{A}{B} = \frac{C}{D}$ be a proportion in which A, B, C, and D are numbers.

To prove $A D = B C$.

Suggestion 1. What process must be performed upon the ratio $\frac{A}{B}$ to produce the product $A D$?

2. What, then, must be done to the ratio $\frac{C}{D}$? Why?

3. What is the effect?

Therefore —

261. CASE II. *If the terms are geometric magnitudes.*

As stated in this case, the theorem is impossible. However, the statement may be allowed to stand with a proper interpretation, viz., **If the terms of a proportion are geo-**

metric magnitudes, the product of the measures of the means equals the product of the measures of the extremes.

In the proportion $\frac{A}{B}=\frac{C}{D}$, let A, B, C, and D represent geometric magnitudes. Let M be a unit of measure of A and B, and N a unit of measure of C and D. § 200.

Suggestion 1. If M is contained in A s times and in B o times, what does the ratio of A to B equal? Why? § 202.

2. In the same way find the measures of C and D. Find the ratio of C to D in terms of their measures.

3. Compare the ratios of the measures, and apply Case I.

Therefore —

Model.

In a proportion, the product of the means equals the product of the extremes.

CASE I. *If all the terms are numbers.*

Let $\frac{A}{B}=\frac{C}{D}$ be a proportion in which A, B, C, and D are numbers.

To prove $A\,D=B\,C$.

$\frac{A}{B}\times B\,D=A\,D$, and $\frac{C}{D}\times B\,D=B\,C$.

$\therefore A\,D=B\,C$. *Ax.* 4.

CASE II. *If the terms are geometric magnitudes.*

Let $\frac{A}{B}=\frac{C}{D}$ be a proportion in which A, B, C, and D are geometric magnitudes. Let M be a unit of measure for A and B, and N a unit of measure for C and D.

To prove $A\,D = B\,C$ with the above interpretation.

If A contains M s times, and B contains it o times, then $\frac{A}{B} = \frac{s}{o}$. § 202.

If C contains N u times, and D contains it v times, then $\frac{C}{D} = \frac{u}{v}$. $\therefore \frac{s}{o} = \frac{u}{v}$. *Ax.* 1.

$\therefore s\,v = o\,u$. Case I.

s and v are the measures of the extremes, and o and u are the measures of the means. § 200.

Therefore —

In a proportion the product of the means equals the product of the extremes.

262. NOTE.—It frequently occurs that mathematical expressions are used which are intelligible under some interpretations and have no meaning under others. For example, the expression $A\,B$ has a definite meaning, if A and B are both numbers; it also has a definite meaning if A is a pure number and B is a quantity, provided A is considered as the multiplier; but if A and B are both quantities, $A\,B$ has no meaning. Likewise, the expression $\frac{A}{B}$ has a definite meaning if A and B are both numbers; it also has a definite meaning if A is a quantity and B a number; also, if A and B are both quantities of the same kind; but has no meaning if A is a number and B is a quantity, or if A and B are quantities unlike in kind. § 202 (1).

In all the following operations upon ratio, and their applications, care should be taken to interpret the symbols of number or quantity, to see whether the principles underlying the fundamental rules of arithmetic and algebra can be applied. The scholia following the theorems in the theory of proportion suggest the limitations that the principles underlying the operations impose upon the propositions.

Ex. 156. Any straight line drawn through the point of intersection of the diagonals of a parallelogram divides the parallelogram into two parts which are equal in all respects.

PROPOSITION II.

263. Theorem: *If the product of two numbers is equal to the product of two other numbers, the factors of one product may be made the means, and the factors of the other product the extremes of a proportion.*

Let $AB = CD$ be an equation in which A, B, C, and D are numbers.

To prove $\frac{A}{D} = \frac{C}{B}$, *or* $\frac{C}{A} = \frac{B}{D}$.

Suggestion 1. What process must be performed upon AB to produce the fraction $\frac{A}{D}$?

2. See method in § 260.

Therefore —

264. COROLLARY I. *If a, b, c, and d in the equation $ab = cd$ be measures respectively of the quantities M, N, O, and P,*

To prove that $\frac{M}{O} = \frac{P}{N}$.

Suggestion 1. $\frac{a}{c} = \frac{d}{b}$. Why?

2. $\frac{a}{c} = \frac{M}{O}$. Why? $\frac{d}{b} = \frac{P}{N}$. Why? § 202.

3. $\frac{M}{O} = \frac{P}{N}$. Why?

Therefore —

265. A **mean proportion** is a proportion in which the means are the same or identical numbers or quantities.

266. A **mean proportional** between two numbers or quantities is the second or third term of a mean proportion.

$\frac{M}{N}=\frac{N}{O}$ is a mean proportion and N is a mean proportional.

267. A **third proportional** is the fourth term of a mean proportion. O is the third proportional, § 266.

PROPOSITION III.

268. Theorem. *A mean proportional equals the square root of the product of the extremes of the proportion.*

Let $\frac{A}{B}=\frac{B}{C}$.

To prove that $B=\sqrt{A\,C}$, in which $A\,C$ means $A\times C$.

CASE I. *If A, B, and C are numbers.*

Suggestion: In the equation $\frac{A}{B}=\frac{B}{C}$, solve algebraically to find the value of B.

Therefore —

CASE II. *If A, B, and C are geometric quantities.*

The theorem is not strictly true as stated, but needs interpretation; viz., the measure of the mean proportional equals the square root of the product of the measures of the extremes of the proportion.

Suggestion: Demonstrate by using method in § 261.

NOTE.—In dealing with proportions, the student should determine whether the terms are numbers or quantities, and observe the laws that govern them.

The expression, "the product of A and B," will be used indiscrimi-

nately, whether A and B are numbers or quantities. because of its brevity. The student must determine from the context whether it is a literal statement of fact, or is used in place of the expression, "the product of the measures of A and B." See § 262.

Proposition IV.

269. Theorem. *If in several successive ratios the consequent of the first is equal to the antecedent of the second, the consequent of the second is equal to the antecedent of the third, etc., then the ratio of the antecedent of the first ratio to the consequent of the last is equal to the product of the ratios.*

Let $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, $\frac{D}{E}$, etc., represent the ratios.

To prove that $\frac{A}{E}$ *is equal to the product of the ratios.*

Case I. *If the terms of the ratios are numbers.*

Suggestion 1. If A, B, C, D, E, etc., are numbers, apply the algebraic rule for multiplying fractions.

Therefore —

Case II. *If the terms of the ratios are geometric magnitudes.*

Suggestion 1. Why must all of the magnitudes be of the same kind? § 202.

Let $\frac{A}{B} = m, \frac{B}{C} = n, \frac{C}{D} = r, \frac{D}{E} = s$; m, n, r, and s being numbers. § 200.

2. Then, $A = m B$, $B = n C$, $C = r D$, $D = s E$. Why? § 270.

3. Since $A = m B$ and $B = n C$, $A = m n C$. Why?
Since $A = m n C$ and $C = r D$, $A = m n r D$.

In the same way $A = m\,n\,r\,s\,E$.

4. Since $A = m\,n\,r\,s\,E$, $\frac{A}{E} = m\,n\,r\,s$. Why?

5. But $m\,n\,r\,s$ is the product of the given ratios.

Therefore —

270. $\frac{A}{B} = m$ means that A contains the unit B, m times.

This is only another form of expression for the statement that A is equal to m times the unit B, which in algebraic language is $A = m\,B$.

To illustrate: $\frac{\text{1 bushel}}{\text{1 peck}} = 4$, is another way of saying 1 bushel = 4 pecks. Page VIII.

NOTE.—A careful interpretation of the symbols should be made at each step in the preceding proposition. The proposition may be illustrated by any problem in reduction descending in denominate numbers; as, to reduce 1 bushel to pints. The table used, expressed in ratio form, is: $\frac{\text{1 bushel}}{\text{1 peck}} = 4$, $\frac{\text{1 peck}}{\text{1 quart}} = 8$, and $\frac{\text{1 quart}}{\text{1 pint}} = 2$. Performing the reduction, 1 bushel $= 4 \times 8 \times 2$ pints $= 64$ pints, or $\frac{\text{1 bushel}}{\text{1 pint}} = 64$.

QUERY: In the equation $A = m\,n\,r\,D$, which symbols represent numbers and which represent quantities?

Ex. 157. Determine a point at a given distance from a given point, and equally distant from two parallel lines.

How many points answer the conditions of the problem? Is this problem always possible?

Ex. 158. Determine a point at a given distance from a given point, and equally distant from two given intersecting lines.

Show when there are four points, when three, when two, when one, and when not any.

PROPOSITION V.

271. Theorem. *Both terms of a ratio can be multiplied by any number without changing the value of the ratio.*

Let $\frac{A}{B}$ represent a ratio.

To prove $\frac{A}{B} = \frac{mA}{mB}$, *m being a number.*

Suggestion 1. Let the ratio $\frac{A}{B} = r$. Then $A =$ what? § 270.

2. Find value of mA. Of $\frac{mA}{mB}$. Give auth.

3. Compare $\frac{mA}{mB}$ and $\frac{A}{B}$.

Therefore —

272. COROLLARY.—*Both terms of a ratio can be divided by the same number without changing the value of the ratio; also, both terms can be divided by the same quantity, provided it is a quantity of the same kind as the terms of the given ratio.*

273. SCHOLIUM.—In the foregoing proposition A and B may both be numbers, or may be like quantities.

Ex. 159. If one circle is inscribed in a right triangle, and another circle circumscribed about the same right triangle, the sum of the diameters of the circles is equal to the sum of the two legs of the right triangle.

Ex. 160. What is the locus of the middle points of all the chords of a circle which are parallel to a given line?
Ex. 116.

PROPOSITION VI.

274. Theorem. *If four numbers, or like quantities, are in proportion, the ratio of the first to the third equals the ratio of the second to the fourth.*

Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.

To prove $\frac{A}{C} = \frac{B}{D}$.

Suggestion 1. Let $\frac{A}{B} = m$. What does $\frac{C}{D} = ?$

2. What is the value of A? Of C?

3. Find the value of $\frac{A}{C}$ and reduce to simplest form.

Therefore —

275. SCHOLIUM.—Proposition VI can be applied only when the four terms are numbers or like quantities. For instance, $\frac{5 \text{ rods}}{10 \text{ rods}}$ may equal $\frac{8 \text{ lbs.}}{16 \text{ lbs.}}$ but to apply Proposition VI would necessitate the unthinkable expression $\frac{5 \text{ rods}}{8 \text{ lbs.}} = \frac{10 \text{ rods}}{16 \text{ lbs.}}$. The definition of ratio is violated, hence the expression is not a proportion.

276. If from any proportion a new proportion is obtained by taking the antecedents for one ratio, and the consequents in the same order for the other ratio, the second proportion is deduced from the first by **alternation.**

$\frac{M}{N} = \frac{R}{S}$ is deduced by alternation from $\frac{M}{R} = \frac{N}{S}$.

Theorem VI may now be stated in the more convenient form, *If four quantities are in proportion, they are in proportion by alternation.*

PROPOSITION VII.

277. **Theorem.** *If four numbers or quantities are in proportion, the ratio of the second to the first equals the ratio of the fourth to the third.*

Let $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{C}}{\mathbf{D}}$ be a given proportion.

To prove $\frac{B}{A} = \frac{D}{C}$.

Suggestion 1. Let $\frac{A}{B} = m$. Then $\frac{C}{D} = m$. Why?

2. From the equations in suggestion 1, find the values of A and C.

3. Divide by A and C, respectively, both members of the two equations obtained in suggestion 2. Compare the results and reduce.

ANOTHER METHOD.

Compare the reciprocals of $\frac{A}{B}$ and $\frac{C}{D}$.

QUERY: Are any limitations to be placed upon the application of Proposition VII?

278. If from any proportion a new proportion is formed by inverting the ratios, the second proportion is deduced from the first by **inversion.**

Theorem VII may now be stated in the more convenient form: "If four quantities are in proportion, they are in proportion by inversion."

PROPOSITION VIII.

279. Theorem. *If four numbers or quantities are in proportion, the ratio of the first plus the second to the second equals the ratio of the third plus the fourth to the fourth.*

Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.

To prove $\frac{A+B}{B} = \frac{C+D}{D}$.

Suggestion 1. Let $\frac{A}{B} = m$. Tnen $\frac{C}{D} =$ what?

2. Find value of $\frac{A+B}{B}$; *i. e.*, divide $A + B$ by B, ($A \div B = m$. Sug. 1).

3. Find value of $\frac{C+D}{D}$. Sug. 1.

4. Compare answers to sugs. 2 and 3.

Therefore —

Or, add 1 to each member of $\frac{A}{B} = \frac{C}{D}$.

280. If from the proportion $\frac{A}{B} = \frac{C}{D}$, the proportion $\frac{A+B}{B} = \frac{C+D}{D}$ is obtained, the second is deduced from the first by **composition.**

Theorem VIII may now be stated in the more convenient form: "*If four quantities are in proportion they are in proportion by composition.*"

PROPOSITION IX.

281. Theorem. ***If four numbers or quantities are in proportion, the ratio of the first minus the second to the second equals the ratio of the third minus the fourth to the fourth.***

Let $\frac{A}{B}=\frac{C}{D}$ be a given proportion.

To prove that $\frac{A-B}{B}=\frac{C-D}{D}$.

Suggestion 1. Let $\frac{A}{B}=m$. Then $\frac{C}{D}=$what?

2. Find value of $\frac{A-B}{B}$; *i. e.*, divide $A-B$ by B.

3. What does $\frac{C-D}{D}$ equal?

4. Compare answers to suggestions 2 and 3.

Therefore —

Or, subtract 1 from each member of the equation $\frac{A}{B}=\frac{C}{D}$.

QUERY: May the terms of one ratio have a different unit of measure from the terms of the other ratio in Propositions VIII and IX? Why?

282. If from the proportion $\frac{A}{B}=\frac{C}{D}$, the proportion $\frac{A-B}{B}=\frac{C-D}{D}$ is obtained, the second is deduced from the first by **division.**

Theorem IX may now be stated in the more convenient form: If four quantities are in proportion, they are in proportion by **division.**

May the proportion $\frac{9^\circ}{6^\circ} = \frac{3 \text{ ft.}}{2 \text{ ft.}}$ be taken by composition or division? By inversion? By alternation?

283. A **continued proportion** is an equality of several ratios.

PROPOSITION X.

284. Theorem. *In a continued proportion, all of whose terms are numbers, or quantities of the same kind, the ratio of the sum of the antecedents to the sum of the consequents equals any of the ratios.*

Let $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{C}}{\mathbf{D}} = \frac{\mathbf{E}}{\mathbf{F}} = \frac{\mathbf{G}}{\mathbf{H}}$, **etc.**

To prove $\frac{A+C+E+G, \text{ etc.},}{B+D+F+H, \text{ etc.},} = \frac{A}{B}$ *or* $\frac{C}{D}$, *etc.*

Suggestion 1. Let each of the given ratios $= x$, and find the value of each antecedent.

2. Find the value of the sum of all the antecedents.

3. From the result of suggestion 2, find the value of x.

4. Compare the value of x, just found, with any of the given ratios.

Therefore —

Ex. 161. If a circle is inscribed in a right triangle the sum of the two legs of the triangle exceeds the hypotenuse by an amount equal to the diameter of the circle.

PROPOSITION XI.

285. Theorem. *The squares of the terms of a proportion are in proportion.*

Let $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{C}}{\mathbf{D}}$.

To prove that $\frac{A^2}{B^2} = \frac{C^2}{D^2}$.

286. COROLLARY I.—*Like powers of the terms of a proportion are in proportion.*

287. COROLLARY II.—*Like roots of the terms of a proportion are in proportion.*

Ex. 162. If $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A}{A-B} = \frac{C}{C-D}$, and also that $\frac{A-B}{A} = \frac{C-D}{C}$.

In deducing the second proportion from the first, are A, B, C, and D restricted to being numbers, or may they be quantities as well; and, if quantities, are they all quantities of the same kind, or may they differ in kind?

Ex. 163. The circle inscribed in an equilateral triangle has the same center as the circle circumscribed about the triangle, and the radius of the circumscribed circle is double that of the inscribed circle.

Ex. 164. If an isosceles triangle is inscribed in a circle prove that the bisector of the vertical angle passes through the center of the circle.

PROPOSITION XII.

288. Theorem. *If three terms of one proportion are respectively equal to the corresponding terms of another proportion, the fourth terms are equal.*

Let $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{C}}{\mathbf{D}}$, and $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{M}}{\mathbf{D}}$.

To prove that $C=M$.

Suggestion 1. Compare $\frac{C}{D}$ and $\frac{M}{D}$.

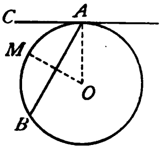
2. Compare C and M.

Therefore —

NOTE.—In all the preceding propositions in the theory of proportion, the theorems are true as stated, if the terms of the proportions are numbers.

If the terms are geometric magnitudes, or quantities, the limitations should be carefully studied and applied. Read § 198–202 (1) on measurement, and note their bearing on the theory of proportion.

Ex. 165. Prove that an angle formed by a tangent and a chord, is measured by one-half the intercepted arc, using the following construction: Drop a ⊥ from the center of ⊙ O to the chord and extend to the arc, as at M. Connect O and A.



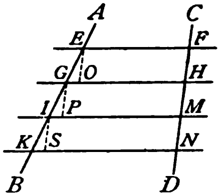
Suggestion: Compare $\angle MOA$ with $\angle BAC$.

Ex. 166. The median of a trapezoid bisects the diagonals, and any line whose extremities are in the bases of the trapezoid.

Ex. 167. Take the proportion $\frac{m}{n} = \frac{o}{s}$ by composition; by inversion; by alternation; by division; by inversion and composition.

PROPOSITION XIII.

289. Theorem. *If parallel lines intercept equal segments on one transversal, they intercept equal segments on all transversals.*



Let E F, G H, etc., represent ‖ lines intercepting equal segments E G, G I, etc., on the transversal A B, and the segments F H, H M, etc., on the transversal C D.

To prove F H equals H M, etc.

Suggestion 1. From E, G, etc., draw lines $E O$, $G P$, etc., ‖ to $C D$.

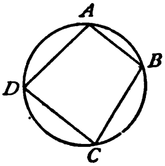
2. Compare EO and FH, GP and HM, etc. Give auth.

3. Compare $E O$, $G P$, etc.; $E O$ and $F H$; $G P$ and $H M$, etc. Give auth.

4 Compare $F H$, $H M$, etc.

Therefore —

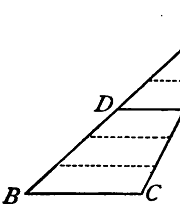
Ex. 168. If a quadrilateral is inscribed in a circle, the sum of the opposite angles equals two right angles. Prove that ∠ A plus ∠ C equals two right ∠s.



Ex. 169. A circle is described on one of the sides of an equilateral triangle as a diameter. Prove that the circumference bisects each of the other two sides.

PROPOSITION XIV.

290. Theorem. *If a line is parallel to the base of a triangle, the ratio of the segments of one side equals the ratio of the segments of the other side.*



Let D E be a line parallel to the base B C of the triangle A B C.

To prove $\frac{AD}{DB} = \frac{AE}{EC}$.

CASE I. *When A D and D B are commensurable.*

Suggestion 1. Measure $A\ D$ and $D\ B$ by a common unit. What is the ratio of $A\ D$ to $D\ B$?

2. Through the points of division in $A\ D$ and $D\ B$ draw lines ∥ to $B\ C$ cutting $A\ C$.

3. Compare the size of the segments in $A\ E$ and $E\ C$. § 289.

4. Compare the number of segments in $A\ E$ and $E\ C$ with those in $A\ D$ and $D\ B$, respectively.

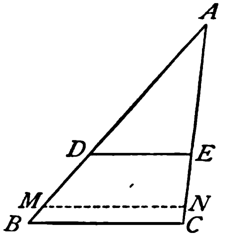
5. What is the ratio of $A\ E$ to $E\ C$? § 202.

6. Compare $\frac{AD}{DB}$ and $\frac{AE}{EC}$. Give auth.

Therefore—

Apply carefully the definition of ratio in Suggestion 5.

CASE II. *When A D and D B are incommensurable.*



Suggestion 1. Divide AD into any number of equal parts, and lay off the unit of measure upon DB as many times as possible. There will be a remainder, MB, less than the unit of measure. Why?

2. Through M draw $MN \parallel$ to BC.

3. Compare $\frac{DM}{AD}$ with $\frac{EN}{AE}$. Give auth.

4. If the unit of measure be continually diminished, the ratio $\frac{DM}{AD}$ is a variable. Why?

5. What is its limit? Why? § 214.

6. The ratio $\frac{EN}{AE}$ is also a variable. What is its limit?

7. As the unit of measure decreases, how do the ratios $\frac{DM}{AD}$ and $\frac{EN}{AE}$ always compare? Sug. 3.

8. How do the ratios $\frac{DB}{DA}$ and $\frac{EC}{EA}$ compare? § 213.

Therefore—

NOTE.—Compare this demonstration with that of § 215.

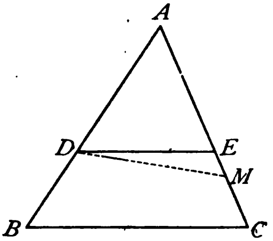
291. COROLLARY. *If a line be drawn parallel to the base of a triangle, one side is to either of its segments as the other side is to its corresponding segment.*

To prove $\frac{AB}{AD} = \frac{AC}{AE}$ or $\frac{AB}{DB} = \frac{AC}{EC}$.

Suggestion. Compare these proportions with the proportion in the conclusion of the proposition, and see by what authority in the theory of proportion the former can be deduced from the latter.

PROPOSITION XV.

292. Theorem. *If a line divides two sides of a triangle proportionally, it is parallel to the base.*



Let A B C be a triangle, and let D E divide the sides, so that $\frac{\mathbf{AD}}{\mathbf{DB}} = \frac{\mathbf{AE}}{\mathbf{EC}}$.

To prove that D E is parallel to B C.

Suggestion 1. Take the proportion of the hypothesis by composition.

2. From D draw $D M \parallel$ to $B C$, and compare $\frac{A B}{D B}$ with $\frac{A C}{M C}$. § 291.

3. In the proportions of Suggestions 1 and 2, compare $E C$ and $M C$. § 288.

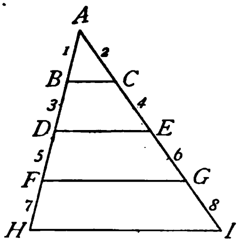
Complete the demonstration.

Therefore—

ANOTHER METHOD. Take the hypothesis $\frac{A D}{A B} = \frac{A E}{A C}$ and work out a demonstration.

PROPOSITION XVI.

293. **Theorem.** *If several lines are drawn parallel to the base of a triangle intersecting the sides, the corresponding segments of the sides form a continued proportion.*



Let A H I represent a triangle, and B C, D E, etc., lines ∥ to the base and intercepting the segments 1, 3, 2, 4, etc., on the sides A H and A I, respectively.

To prove $\frac{1}{2} = \frac{3}{4} = \frac{5}{6}$, *etc.*

Suggestion 1. Compare $\frac{1}{2}$ with $\frac{3}{4}$. § 290 and § 274.

2. Compare $\frac{3}{4}$ with $\frac{AD}{AE}$; (§ 291). $\frac{5}{6}$ with $\frac{AD}{AE}$; (§ 274 and § 290;) $\frac{3}{4}$ with $\frac{5}{6}$ (Axiom 1).

3. Compare $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, and extend the series.

Therefore—

Ex. 170. What has been done to the proportion $\frac{m}{n} = \frac{o}{s}$ to produce $\frac{m+o}{o} = \frac{n+s}{s}$? $\frac{m+n}{m} = \frac{o+s}{o}$? $\frac{m+n}{o+s} = \frac{n}{s}$?

Similar Polygons.

294. Similar Polygons are polygons which are mutually equiangular and have their corresponding sides proportional.

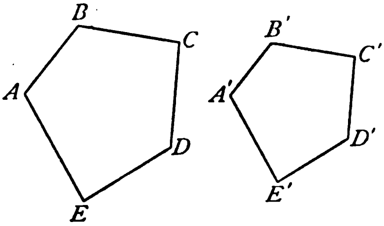
295. Points, lines, and angles, of similar polygons which are similarly situated, are **homologous.**

296. The **ratio of similitude** of similar polygons is the ratio of any two homologous sides.

297. Corollary. *From the definition it follows that in similar polygons, corresponding angles are equal and homologous sides are proportional.*

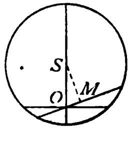
The polygons $A\ B\ C$, etc., and $A'\ B'\ C'$, etc., are similar, if $\angle A = \angle A'$, $\angle B = B'$, etc., and $\frac{A\ B}{A'\ B'} = \frac{B\ C}{B'\ C'} = \frac{C\ D}{C'\ D'}$, etc.

The ratio $\frac{A\ B}{A'\ B'}$ or $\frac{B\ C}{B'\ C'}$, etc., is the **ratio of similitude** of the polygons.



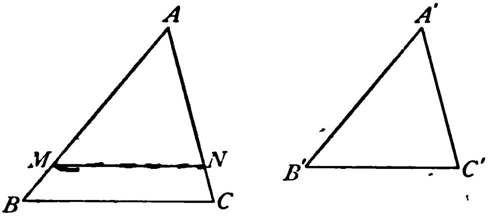
Ex. 171. The locus of the middle points of all chords which pass through a given point, is a circle whose diameter is the line connecting the given point and the center of the circle.

Prove that the circle described on $O\ S$ is the required locus.



Proposition XVII.

298. Theorem. *Two triangles which are mutually equiangular are similar.*



Let ABC and A′B′C′ be two triangles in which the angle A is equal to the angle A′, the angle B is equal to the angle B′, and the angle C is equal to the angle C′.

To prove that the triangles $A\ B\ C$ and $A'\ B'\ C'$ are *similar.*

Suggestion 1. What part of the definition of similar triangles remains to be proved? § 294.

2. Place $\triangle\ A'\ B'\ C'$ upon $\triangle\ A\ B\ C$, so that $A'\ B'$ lies upon $A\ B$, with A' upon A, and B' upon $A\ B$, or $A\ B$ extended, as at M.

3. What direction does $A'\ C'$ take? Why? Let C' fall at some point, as at N.

4. $M\ N$ is $\parallel$ $B\ C$. Why?

5. $\dfrac{A\ B}{A\ M} = \dfrac{A\ C}{A\ N}$. (§ 291.) Hence $\dfrac{A\ B}{A'\ B'} = \dfrac{A\ C}{A'\ C'}$.

6. What remains yet to be proved?

7. Again place $\triangle\ B'\ A'\ C'$ upon $\triangle\ B\ A\ C$, B' upon B, $B'\ A'$ upon $B\ A$, etc. Compare the ratios $\dfrac{A\ B}{A'\ B'}$ with $\dfrac{B\ C}{B'\ C'}$.

8. Compare $\dfrac{A\ B}{A'\ B'}$, $\dfrac{B\ C}{B'\ C'}$, and $\dfrac{A\ C}{A'\ C'}$. Give auth.

Therefore—

299. Corollary I. *If two triangles have two angles of one equal, respectively, to two angles of the other, the triangles are similar.*

300. Corollary II. *Two right triangles which have an acute angle of one equal to an acute angle of the other, are similar.*

301. Scholium. In similar polygons homologous sides are opposite equal angles, and equal angles are opposite homologous sides.

Ex. 172. Through one of the points common to two intersecting circumferences, draw the diameters of the circles and prove that the line connecting the other extremities of the diameters passes through the other point of intersection of the circumferences.

Ex. 173. The bisectors of all angles inscribed in the same segment pass through a common point.

Ex. 174. Draw any two equal chords of a circle that do not intersect. Connect their adjacent extremities. Prove the connecting chords are parallel.

Ex. 175. Draw any two parallel chords, connect their extremities. Prove the connecting chords are equal.

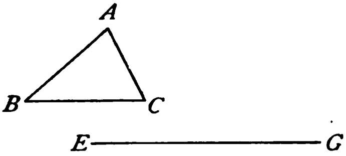
Ex. 176. Draw a tangent to a given circle that shall make a given angle with a given line.

Ex. 177. Given a triangle $A\ B\ C$, how could you use § 299 to construct a triangle similar to $A\ B\ C$ upon a given line M homologous to $A\ C$ of the triangle? Homologous to $A\ B$ of the triangle?

Ex. 178. If triangle $A\ B\ C$, Ex. 177, is a right triangle, the right angle at C, solve the Exercise. § 300.

PROPOSITION XVIII.

302. Problem. *To construct upon a given line, a triangle similar to a given triangle.*



Let A B C be a given triangle and E G a given line.

To construct upon E G, a triangle similar to triangle A B C, E G being homologous to B C.

Let X be the required vertex.

Suggestion 1. What must be true that the required $\triangle\ EGX$ may be similar to ABC? § 299.

2. Make the required construction.

Ex. 179. If you take a pole as long as you are tall, lie upon your back with the pole upright at your feet, and sight over the top of the pole to some adjacent object, how high is the object?

What measurement must you take to find out?

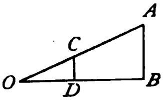
Ex. 180. How could you measure the height of a church spire, a tree or a building? How could you use shadows to find heights of objects?

Ex. 181. Determine as accurately as possible the height of the gable of your school building.

NOTE.—Not over two should work together, and answers should be compared in class.

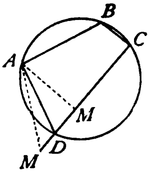
Ex. 182. To measure height *A B*.

Suggestion. Set up a pole perpendicularly at a convenient point, as at D. From O sight to A. Let another person move a card upon the pole until a point in line AO is determined on the pole. Make the required measurements and determine the height AB. § 298.



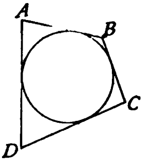
NOTE.—The line O B may be the line of the ground or some other convenient line.

Ex. 183. If the sum of the opposite angles of a quadrilateral equals two right angles, prove that a circumference is possible through the four vertices.

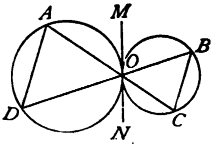


Suggestion. A circumference can be described through three vertices, as A, B and C. If it does not pass through D, it must cut the line CD, or CD extended, as at M. Compare the sum of $\angle$s B and CMA with two right $\angle$s. Also, the sum of $\angle$s B and CDA with two right $\angle$s.

Ex. 184. If a quadrilateral is circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.



Ex. 185. If two circles are tangent, and two secants are drawn through the point of contact, the chords joining the intersections of the secants and the circumferences are parallel.

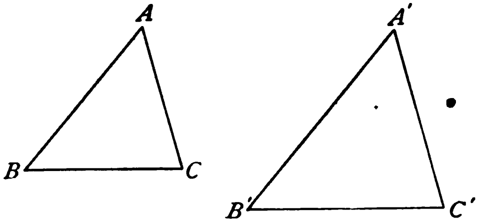


Prove AD parallel to BC.

Suggestion. Draw the common tangent, MN. Compare $\angle AOM$ with $\angle CON$. Compare $\angle AOM$ with $\angle ADO$; also, $\angle CON$ with $\angle OBC$.

PROPOSITION XIX.

303. Theorem. *If two triangles have an angle of one equal to an angle of the other, and the sides including the equal angles proportional, the triangles are similar.*



In the triangles A B C and A′ B′ C′, let the angle A equal the angle A′ and let the ratio $\frac{AB}{A'B'}$ equal the ratio $\frac{AC}{A'C'}$.

To prove that the triangles A B C and A′ B′ C′ are similar.

Suggestion 1. What must be proved in addition to the hypothesis to make the triangles similar according to the definition?

2. Place the △ *A B C* upon the △ *A′ B′ C′*, with *A* upon *A′*, and *A B* and *A C* upon *A′ B′* and *A′ C′*, respectively. Why is this possible?

3. Where do *B* and *C* fall?

4. *B C* is ∥ to *B′ C′*. Why? § 292.

5. How do the ∠s *B* and *C* compare with the ∠s *B′* and *C′*, respectively? Why?

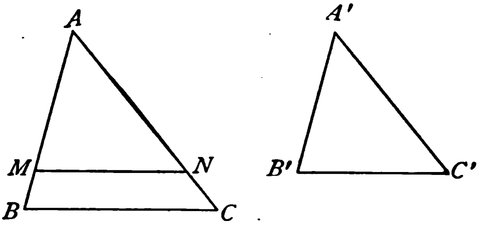
6. Complete the demonstration.

Therefore—

Ex. 186. Let *A B C* and *A′ B′ C′* be two similar triangles. *A B* = 7 feet, *A′ B′* = 10 feet, *A C* = 14 feet. Find the length of *A′ C′*. If *A B* is 13 feet, *A′ B′* and *A C* 10 and 11 feet, respectively, find the length of *A′ C′*.

PROPOSITION XX.

304. Theorem. *Two triangles whose corresponding sides are proportional are similar.*



Let ABC and A'B'C' be two triangles such that $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$.

To prove that triangles A B C and A' B' C' are similar.

Suggestion 1. What remains to be proved according to the definition of similar triangles?

2. Upon $A\ B$ lay off $A\ M$ equal to $A'\ B'$, and upon $A\ C$ lay off $A\ N$ equal to $A'\ C'$; connect M and N. Compare △s $A\ M\ N$ and $A\ B\ C$. § 303.

3. Compare the ratios $\frac{A\ M}{A\ B}$ with $\frac{M\ N}{B\ C}$; also $\frac{A'\ B'}{A\ B}$ and $\frac{B'\ C'}{B\ C}$.

4. Compare $M\ N$ and $B'\ C'$; give auth. (§ 288); △s $A\ M\ N$ and $A'\ B'\ C'$. Give auth.

5. Compare △ $A\ B\ C$ and △ $A'\ B'\ C'$.

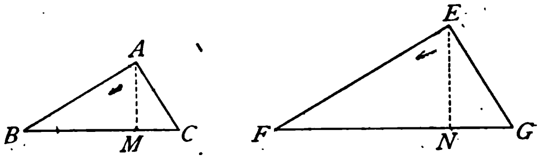
Therefore—

Ex. 187. The sides of triangle $A\ B\ C$ are, respectively, 4, 8, and 11 feet. In a similar triangle, $A'\ B'\ C'$, the side homologous to 4, is 6 feet. Find the other two sides of triangle $A'\ B'\ C'$.

Give a summary of the tests for similarity of triangles.

PROPOSITION XXI.

305. Theorem. *The ratio of homologous altitudes of similar triangles is equal to the ratio of similitude of the triangles.*



Let AM and EN be homologous altitudes in the similar triangles ABC and EFG, and let $\frac{AB}{EF}$ represent the ratio of similitude of the triangles.

To prove that $\frac{AM}{EN} = \frac{AB}{EF}$.

Suggestion 1. Compare △s AMB and ENF. § 300. Complete the demonstration.

306. COROLLARY. *In similar triangles homologous altitudes are proportional to the bases.*

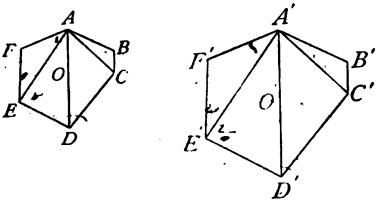
To prove $\frac{AM}{EN} = \frac{BC}{FG}$.

Ex. 188. Through a given point, to draw a straight line so that the portion of it intercepted between two given intersecting lines is divided at the point into two equal parts.

Suggestion. Through the point draw a line parallel to one of the given lines. § 290.

PROPOSITION XXII.

307. Theorem. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



Let the polygons ABC, etc., and A′B′C′, etc., be composed of the same number of triangles, similar each to each and similarly placed, ABC being similar to A′B′C′, etc.

To prove that the polygons are similar.

Suggestion 1. What must be proved to know that the polygons are similar? § 294.

2. Compare ∠ B with ∠ B', ∠ C with ∠ C', ∠ D with ∠ D', etc. Give auth.

3. Compare the ratio $\frac{AB}{A'B'}$ with $\frac{BC}{B'C'}$. Give auth.

4. Compare the ratio $\frac{BC}{B'C'}$ with $\frac{AC}{A'C'}$; also the ratio $\frac{CD}{C'D'}$ with $\frac{AC}{A'C'}$. How, then, does $\frac{BC}{B'C'}$ compare with $\frac{CD}{C'D'}$?

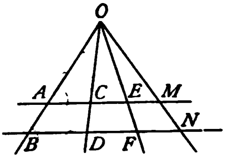
5. In a similar manner compare the ratios of other corresponding sides in the two polygons.

6. Apply the definition of similar polygons. Sug. 1.

Therefore—

308. In this early study of the subject, in stating a proportion deduced from two similar polygons, homologous sides should be selected for the terms of each ratio; the antecedents of the ratios should be taken from one of the polygons and the consequents from the other.

If any other form of the proportion is desired, it can be modified by the appropriate proposition in the theory of proportion. For instance, if triangle I is similar to triangle II, and A is homologous to A', etc., the proportion $\frac{A}{A'} = \frac{B}{B'}$ may be taken. (Apply above test.) If the proportion, $\frac{A}{B} = \frac{A'}{B'}$, is desired, the original proportion can be taken by alternation. Later, when the pupil is sure of himself, the work can be abbreviated, and any desired proportion taken at once.



Ex. 189. Straight lines drawn through any point intercept proportional segments upon two parallel lines.

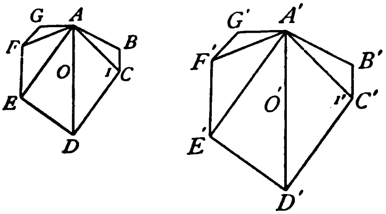
Prove that $\frac{AC}{BD} = \frac{CE}{DF} = \frac{EM}{FN}$, etc.

Suggestion. Compare each of the ratios $\frac{AC}{BD}$ and $\frac{CE}{DF}$ with the ratio $\frac{OC}{OD}$. Complete the demonstration.

Ex. 190. Upon a given base construct a right triangle, having given the perpendicular from the right angle to the hypotenuse.

PROPOSITION XXIII.

309. Theorem. CONVERSE OF PROPOSITION XXII. *Two similar polygons can be divided into the same number of triangles, similar each to each and similarly placed.*



Let polygons ABC, etc., and A′B′C′, etc., be similar, and let all possible diagonals be drawn from two corresponding vertices, as A and A′.

To prove that the triangles in one polygon are similar, respectively, to those in the other polygon.

Suggestion 1. What must be proved to establish the theorem? § 298, § 303 or § 304.

2. Compare $\triangle\ A\ B\ C$ and $\triangle\ A'\ B'\ C'$.

3. Compare the ratios $\frac{B\,C}{B'\,C'}$ with $\frac{A\,C}{A'\,C'}$; $\frac{B\,C}{B'\,C'}$ with $\frac{C\,D}{C'\,D'}$. Give auth.

Then, how do $\frac{A\,C}{A'\,C'}$ and $\frac{C\,D}{C'\,D'}$ compare? Give auth.

4. Compare $\angle\ 1$ and $\angle\ 1'$.

5. Compare $\triangle$s $A\ D\ C$ and $A'\ D'\ C'$. Give auth.

6. In a similar manner, compare the other pairs of $\triangle$s. Complete the demonstration.

Therefore—

NOTE.—It would be profitable to compare the demonstrations in § 307 and § 309.

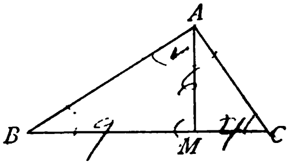
PROPOSITION XXIV.

310. Theorem. *If a perpendicular be dropped to the hypotenuse from the vertex of the right angle in a right triangle:*

PART I. *The triangles thus formed are similar to each other and to the whole triangle.*

PART II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

PART III. *Each side is a mean proportional between the hypotenuse and the segment adjacent to that side.*



Let ABC represent a right triangle, the right angle at A, and AM the perpendicular drawn from A to the hypotenuse BC.

PART I. To prove that the triangles AMC, AMB, and BAC are similar.

Suggestion. What are the tests for similarity of △s? (§§ 298, 303, and 304.) Select the one suited to this case.

PART II. *To prove* $\frac{BM}{AM} = \frac{AM}{MC}$. (See Ex. 119.)

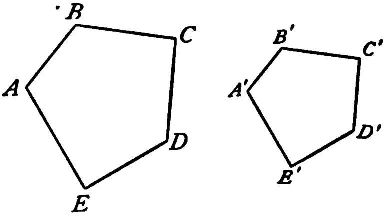
Suggestion. Select two △s, one of which contains the two antecedents and the other the two consequents, and determine the required proportion. § 308.

PART III. *To prove that* $\frac{CB}{AB} = \frac{AB}{BM}$; or, *that* $\frac{CB}{CA} = \frac{CA}{CM}$.

Suggestion. See Sug. Part II.

PROPOSITION XXV.

311. Theorem. *The ratio of the perimeters of two similar polygons equals the ratio of similitude of the polygons.*



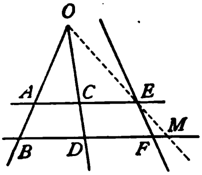
Let polygons A D and A′D′ be similar.

To prove that the ratio, $\dfrac{\text{perimeter } ABC\text{, etc.,}}{\text{perimeter } A'B'C'\text{, etc.,}}$, *is equal to the ratio of similitude* $\dfrac{AB}{A'B'}$, *or* $\dfrac{BC}{B'C'}$, *etc.*

Suggestion. § 284.

Therefore—

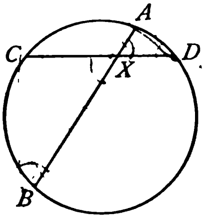
Ex. 187A. If two parallel lines are cut proportionally by a set of secant lines, prove that the secant lines pass through a common point. If $\dfrac{AC}{BD} = \dfrac{CE}{DF}$, etc., prove that the lines BA, DC, FE, etc., intersect at a common point.



Suggestion. Extend two of them, as BA and DC, until they meet at O. Connect O and E, and extend OE to the other of the parallels at M. Compare EM and EF.

PROPOSITION XXVI.

312. Theorem. *If two chords of a circle intersect, the ratio of either segment of the first to either segment of the second is equal to the ratio of the remaining segment of the second to the remaining segment of the first.*



Let the chords A B and C D intersect at X.

To prove that $\frac{AX}{DX} = \frac{CX}{BX}$, *or* $\frac{AX}{CX} = \frac{DX}{BX}$.

Suggestion 1. As no $\triangle$s are given in the theorem, two $\triangle$s must be constructed that contain, one the required antecedents, and the other the required consequents. Auth.

2. Prove the constructed $\triangle$s are similar.

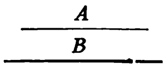
3. Establish the required proportion.

Therefore—

Establish both proportions in the special enunciation.

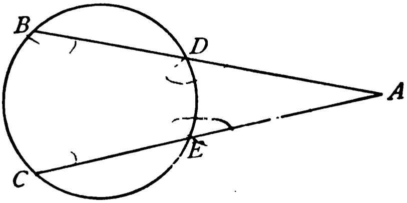
Ex. 188A. A perpendicular dropped from the circumference of a circle upon the diameter is a mean proportional between the segments of the diameter.

Ex. 189A. Use Ex. 188A to construct a mean proportional to the lines *A* and *B*.



PROPOSITION XXVII.

313. Theorem. *If two secants intersect without a circle, the ratio of the first secant to the second is equal to the ratio of the external segment of the second to the external segment of the first.*



Let A B and A C represent two secants meeting at A and intersecting the circle in B, D, E, and C.

To prove that $\frac{AB}{AC} = \frac{AE}{AD}$.

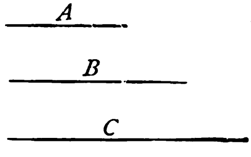
Suggestion. Make two △s, by drawing construction lines, according to the principle laid down in Suggestion 1, § 312.

Complete the demonstration.

Ex. 190A. By § 312 find a line X, that is a fourth proportional to A, B and C; *i. e.*, if $\frac{A}{B} = \frac{C}{X}$, find X.

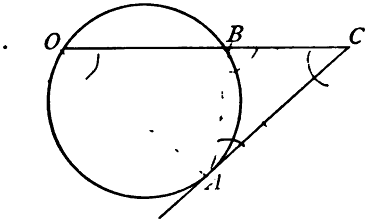
§ 259.

Suggestion. If the lines A, B, and C are located as suggested by the theorem, three points are fixed by which the circle can be constructed. Then X can be found.



PROPOSITION XXVIII.

314. Theorem. *If a secant and a tangent meet without a circle, the tangent is a mean proportional between the secant and its external segment.*



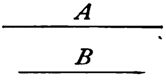
Let AC represent a tangent, and OC a secant meeting the tangent at the point C.

To prove that $\frac{OC}{AC} = \frac{AC}{BC}$.

Suggestion. Construct △s, so that the antecedents are in one △ and the consequents in another.

By the method suggested in § 312, complete the demonstration.

Ex. 191. A mean proportional between two lines can be constructed by this theorem, § 314. Try it.



$\frac{A}{X} = \frac{X}{B}$. Find X.

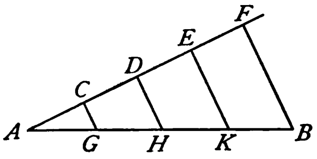
Ex. 192. Construct a fourth proportional to three given lines by § 313. (Sug. for Ex. 190A.)

Ex. 193. Construct a third proportional to two lines. § 310, II or III.

Ex. 194. Construct a mean proportional to *A* and *B* by § 310, Part II. By Part III.

PROPOSITION XXIX.

315. **Problem.** *To divide a given straight line into any number of equal parts.*



Let A B be the given straight line.

To divide A B into any number of equal parts; for example, into four parts.

Suggestion. In what propositions has the truth that a line is divided into equal parts been established?

§ 289, § 293, etc.

Make the construction in accordance with the truth selected.

Further suggestions: *1.* Draw an indefinite line through *A*, making any convenient ∠ with the given line.

2. Lay off upon this line from *A* an estimated fourth of *A B*, four times. Connect the last point of division with *B*. Complete the solution.

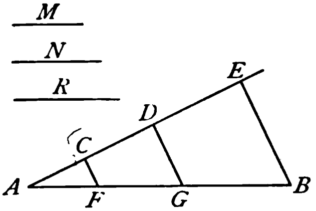
Ex. 195. What is the locus of the vertex of the right angle of a right-angled triangle constructed upon a given line as a hypotenuse? (See § 222.)

Ex. 196. Construct upon a given line, as a chord, a segment of a circle which can contain a given angle.

Suggestion. Construct at the ends of the given line, and upon it, angles whose sum is the supplement of the given angle. Circumscribe a ⊙ about the △ thus formed.

PROPOSITION XXX.

316. Problem. *To divide a given straight line into parts proportional to any given lines.*



Let A B be the given straight line, and M, N, R, etc., be the given lines.

To divide A B into parts such that the first part is to M as the second part is to N, as the third part is to R, etc.

Suggestion 1. In what proposition has the truth concerning the division of a line into several proportional parts been demonstrated? § 293.

Apply the truth in the solution of the problem. (See method, § 315.)

Ex. 197. With a given fixed base, and a given vertical angle of a triangle, find the locus of the vertex. (See Exercise 196.)

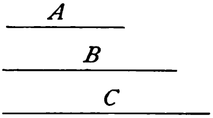
Ex. 198. Construct a triangle whose base and vertical angle are given, and whose vertex is at a given distance from the middle point of the base.

Suggestion. Solve by intersection of two loci.

Ex. 199. If two circles intersect and their common chord be extended, prove that tangents drawn to the two circles from any point in the chord extended are equal.

PROPOSITION XXXI.

317. Problem. *To construct a fourth proportional to three given lines.*



Let A, B, and C be three given lines.

To construct a fourth proportional to A, B, and C.

Suggestion. Employ one of the many propositions from which a proportion of four different terms has been determined. § 290, § 291, § 298, § 303, § 304, § 312, Ex. 189, etc.

Make several solutions.

Illustration of solution, using Exercise 189.

Let A, B, and C be the given lines.

To find X, the fourth proportional to A, B, and C.

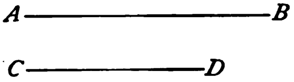
In Exercise 189, $\frac{AC}{BD} = \frac{CE}{DF}$. As our proportion is to read $\frac{A}{B} = \frac{C}{X}$, A, B, C, and X must take the relative positions in the figure occupied, respectively, by AC, BD, CE, and DF.

Since AC and BD are $\parallel$, draw two indefinite $\parallel$ lines, and lay off A and B in the same relative positions as AC and BD. OB and OD of the figure are determined by the ends of AC and BD. Hence, draw lines through ends of A and B and extend until they meet, as at O. They must meet unless A and B are equal. Why? § 151.

C can now be laid off in position of CE, an extension of *A*. The line corresponding to *O E* is now determined by points *O* and the end of *C*. Hence, draw a line through *O* and end of *C*, meeting line *B* extended. *X* is now determined.

PROPOSITION XXXII.

318. Problem. *To construct a third proportional to two given lines.*



Let A B and C D be two given lines.

To construct a third proportional to A B and C D.

QUERY. Can you find a third proportional by means of § 290, § 312, § 313, Exercise 189, etc.? Try it.

Suggestion. Every proposition in which a mean proportional has been established furnishes the basis of a solution also. § 310, II, § 310, III, § 314.

Name all the theorems on which a solution can be based.

Ex. 200. What is the locus of the vertex on one side of the base of a triangle having a given base and a given altitude?

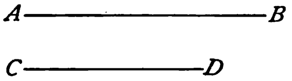
Ex. 201. What is the locus of the vertex of a triangle having a given base and a given altitude?

Ex. 202. Construct a triangle having a given base, a given altitude, and a given vertical angle. Ex. 197.

Suggestion. Solve by intersection of two loci.

PROPOSITION XXXIII.

319. Problem. *To construct a mean proportional to two given lines.*



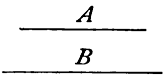
Let A B and C D be two given lines.

To construct a mean proportional to A B and C D.

A solution can be based on § 310, II, 310, III, and § 314.

Ex. 203. Prove Proposition XXI, § 305, by taking $\frac{B\,C}{F\,G}$ as the ratio of similitude of the triangles.

Ex. 204. Construct a line that shall equal the $\sqrt{a\,b}$.



Suggestion 1. To find line x.

Then $x = \sqrt{a\,b}$ (§ 268 and § 268-note) (2) $x\,x = a\,b$; Why? (3) § 264, (4) § 319.

Ex. 205. Construct a line that shall equal $\sqrt{2}$. Suggestion. $\sqrt{2} = \sqrt{2 \times 1}$.

Select a line for a unit.

Ex. 206. Construct a line that shall equal $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$; $\sqrt{10}$; $\sqrt{12}$.

Make two solutions each for $\sqrt{6}$ and $\sqrt{10}$. Make three for $\sqrt{12}$. *Sug.* $\sqrt{6} = \sqrt{2 \times 3}$, or $\sqrt{6 \times 1}$.

Ex. 207. $A\,B$ is a chord of a circle. $C\,D$ is a diameter perpendicular to $A\,B$, and intersecting $A\,B$ at E. $C\,E$ is 5 in. and $A\,C$ 10 in. Find the diameter of the circle.

320. Review and Summary.

1. If $\frac{M}{N} = \frac{S}{O}$, name at sight what changes in the theory of proportion have taken place in the following, and verify your statement:

(*a*) $\frac{N}{M} = \frac{O}{S}$.

(*b*) $\frac{M}{S} = \frac{N}{O}$.

(*c*) $\frac{N}{M+N} = \frac{O}{S+O}$.

(*d*) $\frac{N}{O} = \frac{M+N}{S+O}$.

(*e*) $\frac{M+N}{M} = \frac{S+O}{S}$.

(*f*) $\frac{M+S}{S} = \frac{N+O}{O}$.

(*g*) $\frac{S}{M-S} = \frac{O}{N-O}$.

(*h*) $\frac{M-N}{M} = \frac{S-O}{S}$.

Extend the list.

2. Name all the cases in which a mean proportional has been found.

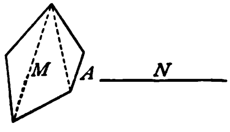
3. Name all the methods by which triangles are found to be similar.

4. Name the methods by which two polygons can be found to be similar.

5. By what theorems can third proportionals be constructed?

6. By what theorems can mean proportionals be constructed?

Ex. 208. Construct a polygon similar to a given polygon upon a given line homologous to a given side of the polygon. Construct a polygon similar to *M* upon line *N* homologous to *A*. §§302, 307.



CHAPTER IV.

COMPARISON AND MEASUREMENT OF POLYGONS.

Definition.

321. The **area** of a surface is its ratio to some selected unit of measure, times the unit of measure. §§ 197 to 202.

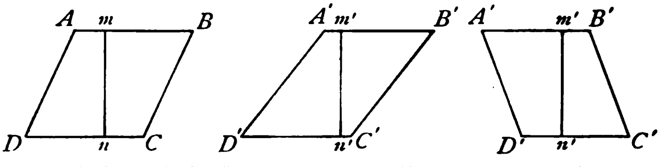
If M denotes the unit of measure for determining the area of surface A, and if the ratio $\frac{A}{M}$ equals b, the area of the surface A is $b\,M$; i. e., b times the unit M.

322. The **unit of measure** for surfaces is a square, whose side is a given linear unit; as a square inch, a square foot, etc.

An acre is the one exception to this definition.

PROPOSITION I.

323. Theorem. *Two parallelograms having equal bases and equal altitudes, are equal in area.*



Let A C and A′ C′ be two parallelograms having the bases D C and D′C′ equal, and the altitudes *m n* and *m′ n′* equal.

To prove that the area of A C is equal to the area of A′ C′.

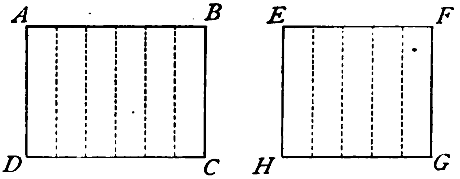
Suggestion. Place ▱ AC upon ▱ $A'C'$, DC upon $D'C'$, and compare the parts external to each other.

Therefore —

Make drawings for the superposition of one parallelogram upon the other in many different forms, but all under the conditions of the theorem.

PROPOSITION II.

324. Theorem. *If two rectangles have equal altitudes, the ratio of their areas is equal to the ratio of their bases.*



Let AC and EG be two rectangles having equal altitudes, DA and HE.

To prove that $\frac{\text{area of } AC}{\text{area of } EG}$ *is equal to* $\frac{DC}{HG}$.

CASE I. *When DC and HG are commensurable.*

Suggestion 1. By thorough preparation on § 215 and § 290, this proposition can be demonstrated as an original exercise.

2. Measure DC and HG by some common unit of measure. Let the unit be contained m times in DC, and n times in HG.

3. What does the ratio of DC to HG equal? Why?

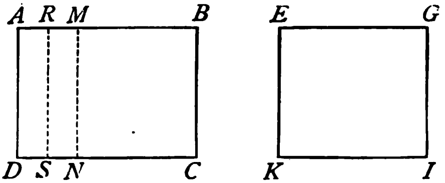
4. At the points of division erect ⊥s to DC and HG, and extend to the secondary bases, thus dividing AC and EG into rectangles. Why rectangles?

5. How do these rectangles compare in size? Why? How many rectangles in $A C$? In $E G$? Why?

6. What does the ratio of the areas of the rectangles $A C$ and $E G$ equal? Why? § 202.

7. Compare the ratio of the areas with the ratio of the bases.

CASE II. *When the bases are incommensurable.*



***Suggestion** 1.* Take any unit of $K I$ and lay it off on $D C$ as many times as possible. There must be a remainder less than one of these parts. Why? Suppose the unit of $K I$ is contained in $D C$ a certain number of times with a remainder $D N$. Erect $N M \perp$ to $D C$, at the point N.

2. Compare the ratio of the areas of $M C$ and $E I$ with the ratio of $N C$ and $K I$.

3. Apply a unit of measure smaller than $D N$. Let the remainder be $D S$. Erect a $\perp$ as before. Compare ratio $\frac{RC}{EI}$ with $\frac{SC}{KI}$. Let the unit continually decrease.

4. The ratio $\frac{RC}{EI}$ is a variable. Why? What is its limit? What is the limit of variable $\frac{SC}{KI}$?

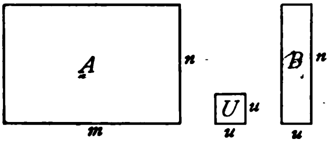
5. Compare the limits.

Therefore —

325. COROLLARY.—*If two rectangles have equal bases, the ratio of their areas is equal to the ratio of their altitudes.*

PROPOSITION III.

326. Theorem. *The number of units of area in any rectangle is equal to the product of the measures of its base and altitude.*



Let A represent a rectangle, U a unit of measure for area, and u the linear unit, viz., a side of the square U. Let u be contained a times in base m, and b times in altitude n; i. e., $\frac{m}{u} = a$ and $\frac{n}{u} = b$.

To prove that $\frac{A}{U}$ *is equal to* $a \times b$, *or that* A *contains the unit* U $a \times b$ *times.*

Suggestion 1. Construct a rectangle B whose altitude is equal to n and whose base is equal to u.

2. What is the ratio of A to B? Of B to U? § 324.

3. What, then, is the ratio of A to U? § 269.

Therefore —

Ex. 209. The ratio of the squares of the legs of a right triangle is equal to the ratio of the segments of the hypotenuse formed by dropping a perpendicular from the vertex of the right angle upon the hypotenuse.

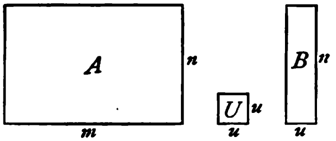
Suggestion 1. See figure in § 310. By use of Part III of the proposition in § 310, find an expression for the square of each leg of the triangle. § 268.

2. What is the ratio of the squares found?

Model.

PROPOSITION III.

Theorem. *The number of units of area in any rectangle is equal to the product of the measures of its base and altitude.*



Let A represent a rectangle, U, a unit of measure for area, and *u* the linear unit, viz., a side of the square U. Let *u* be contained *a* times in base *m* and *b* times in altitude *n*; i, e., let $\frac{m}{u} = a$, and $\frac{n}{u} = b$.

To prove that $\frac{A}{U} = a \times b$, *or that A contains the unit U* $a \times b$ *times.*

Construct rectangle B with base equal to u and altitude equal to n.

$\frac{A}{B} = \frac{m}{u} = a$. §324. $\frac{B}{U} = \frac{n}{u} = b$. §325. $\frac{A}{U} = a \times b$. §269.

Q. E. D.

327. COROLLARY.—*The area of a rectangle is equal to the product of the measures of the base and altitude times the unit of measure.*

Since $\frac{A}{U} = a \times b$, $A = (a \times b)\, U$, i. e., the area of A is $a \times b$ times the unit U.

Illustration: If a rectangle is 6 ft. by 10 ft., the area is (6 × 10) square feet.

328. SCHOLIUM I. In the applications of this theorem the base and altitude must be expressed in terms of the same unit, and the unit of area must be the square whose side is the linear unit.

329. SCHOLIUM II. The expression, "the product of the base and altitude," is a common abbreviation for "the product of the measures of the base and altitude times the unit of area."

(*a*) "The product of two lines," or "the square of a line," when "square" means the algebraic second power, must not be interpreted in any other sense than that just stated. With this interpretation, the Corollary is usually stated as follows: The area of a rectangle is equal to the product of its base and altitude.

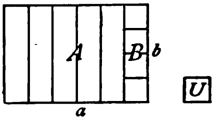
(*b*) The numbers which represent the measures of the lines may be integral, fractional, or incommensurable.

(*c*) The **rectangle of two lines** is an expression which is sometimes used instead of the product of two lines.

NOTE.—Possibly the thought in the foregoing demonstration can be made clearer by an illustration in the arithmetical form of analysis.

(a) Since A contains B a times, and B contains U b times, A must contain U $a \times b$ times.

(b) To take a particular case, let the base and altitude be measured by some linear unit, as 1 inch; and suppose this unit is contained 4 times in the altitude and 6 times in the base; then, as seen in the figure, there are 6 columns, with 4 squares in each column, and hence, in all, 6 x 4 squares; i. e., 24 square inches.



It can be readily seen, by constructing a figure, that if the unit is contained a fractional number of times in the base and altitude the same rule is reached.

QUERY: Can the conclusion of the proposition, in all its generality, be drawn from the method in common use in arithmetic? § 324. Case II.

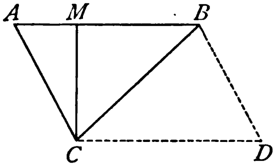
PROPOSITION IV.

330. Theorem. *The area of a parallelogram is equal to the product of its base and altitude.*

Suggestion. § 323 and § 326.

PROPOSITION V.

331. Theorem. *The area of a triangle is equal to one-half the product of its base and altitude.*



Let ABC represent a triangle, AB its base, and MC its altitude.

To prove that the area of the triangle A B C is equal to one-half of the product of A B and M C.

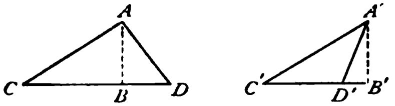
Suggestion 1. Enlarge $\triangle A B C$ so as to make a ▱ with $A B$ and $A C$ as two of its sides. State authority for your construction.

2. Compare the figures $A B C$ and $A B D C$ in respect to area. Compare their bases and altitudes.

3. What is the area of $\triangle A B C$ in terms of its base and altitude?

332. COROLLARY I.—*A triangle is equal in area to one-half a rectangle if it has the same base and altitude.*

333. COROLLARY II.—*If two triangles have equal altitudes their areas have the same ratio as their bases.*



In triangles $A\,C\,D$ and $A'\,C'\,D'$, let the altitudes $A\,B$ and $A'\,B'$ be equal.

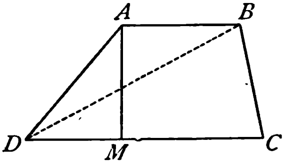
Then $\dfrac{A\,C\,D}{A'\,C'\,D'} = \dfrac{\frac{1}{2}\,A\,B \times C\,D}{\frac{1}{2}A'\,B' \times C'\,D'} = \dfrac{C\,D'}{C'\,D}$. Give auth. (§ 272.)

In a similar manner, prove that if two triangles have equal bases their areas have the same ratio as their altitudes.

334. Scholium.—For the interpretation of the expression, "the product of the base and altitude," in propositions IV and V, see § 329.

Proposition VI.

335. **Theorem.** *The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases.*



Let A B C D represent a trapezoid, and A M its altitude.

To prove that the area of the trapezoid, A B C D, is equal to one-half the product of A M and the sum of A B and D C.

Suggestion 1. Draw the diagonal, $B\,D$.

2. Let $A\,B$ be the base of $\triangle\,D\,A\,B$, $D\,C$ be the base of $\triangle\,D\,C\,B$, and $A\,M$ the altitude of each. Why is $A\,M$ the altitude of each $\triangle$?

3. Find the area of each $\triangle$. Find the area of the trapezoid.

Therefore —

Further suggestions:

Area $DAB = \frac{1}{2}(AM \times AB)$.

Area $DBC = \frac{1}{2}(AM \times DC)$.

Area $DABC = \frac{1}{2}(AM \times AB) + \frac{1}{2}(AM \times DC) = \frac{1}{2}AM(AB + DC)$.

Explain the reductions algebraically.

QUERY: In terms of what other lines can the area of a trapezoid be expressed? Ex. 139.

Area of a Polygon.

336. Various methods have been used to find the area of irregular polygons. Among the methods used the following may be noticed:

From any vertex of the polygon draw all possible diagonals, as in Fig. 1. The polygon is by this means divided into triangles, and if the bases and altitudes of these triangles are measured, their areas can be computed, and by addition the area of the polygon can be found.

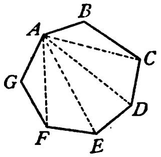


FIG. 1.

Another method is to draw the longest diagonal of the polygon, and from the vertices drop perpendiculars upon this diagonal, as in Fig. 2. The polygon is, in this way, divided into triangles and trapezoids, and if the bases and altitudes of these triangles and trapezoids are measured, their areas can be computed and the area of the polygon obtained.

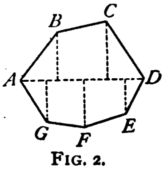


FIG. 2.

Still another method is to draw, through any vertex of the polygon, a straigth line, exterior to the polygon, and from the vertices drop perpendiculars upon this line, as in Fig. 3. In this way, triangles and trapezoids are formed. If the various bases and altitudes are measured, the areas of the triangles and trapezoids can be computed, and if the areas of the parts exterior to the polygon be subtracted from the sum of the other areas, the difference is the area of the polygon.

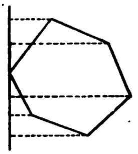


FIG. 3.

The method just described is the one by which surveyors sometimes compute irregular areas bounded by straight lines. If the map or outline of a field, or any irregular polygon, is drawn to scale, any line of the polygon can be determined by means of the homologous line in the constructed polygon. § 294.

Ex. 210. The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.

Suggestion 1. See Sug. 1, of Ex. 209.

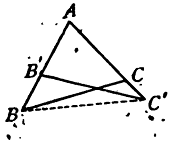
2. Add the squares.

Ex. 211. To construct a triangle similar to a given triangle having a given perimeter.

Ex. 212. If two triangles have an angle of one equal to an angle of the other, the ratio of their areas equals the ratio of the products of the sides including the equal angles.

To prove $\frac{\triangle ABC}{\triangle AB'C'} = \frac{AB \times AC}{AB' \times AC'}$.

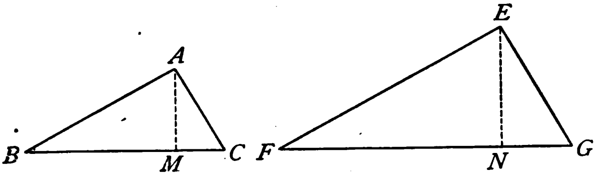
Suggestion. Connect B and C'. Compare each $\triangle$ with ABC'. § 269.



Ex. 213. Prove that the area of a rhombus equals one half the product of its diagonals.

PROPOSITION VII.

337. Theorem. *The ratio of the areas of two similar triangles is equal to the ratio of the squares of their homologous sides, or homologous altitudes.*



Let ABC and EFG represent two similar triangles, and AM and EN homologous altitudes.

To prove $\dfrac{\triangle ABC}{\triangle EFG} = \dfrac{\overline{BC}^2}{\overline{FG}^2}$ *or* $\dfrac{\overline{AM}^2}{\overline{EN}^2}$.

Suggestion 1. The area of $\triangle ABC = \frac{1}{2} BC \times AM$.

2. The area of $\triangle EFG = \frac{1}{2} FG \times EN$.

$$3.\ \therefore \frac{area\ \triangle ABC}{area\ \triangle EFG} = \frac{\frac{1}{2} BC \times AM}{\frac{1}{2} FG \times EN}\ (\S\ 202) =$$

$$\frac{BC \times AM}{FG \times EN} = \frac{BC}{FG} \times \frac{AM}{EN} = \frac{BC}{FG} \times \frac{BC}{FG} \left(\frac{BC}{FG} = \frac{AM}{EN}.\right.$$

$$\left.\text{Why?}\right) = \frac{\overline{BC}^2}{\overline{FG}^2} = \frac{\overline{AM}^2}{\overline{EN}^2}.$$

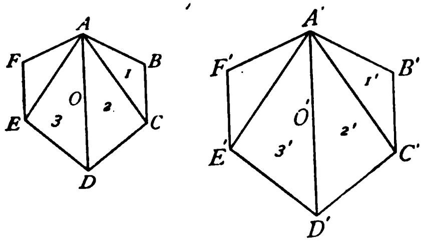
Give authority for each step.

Therefore —

Ex. 214. Given the area of a trapezoid equal to 104 sq. ft., the altitude equal to 6 ft., and the difference between the bases equal to 2 ft. Find the two bases of the trapezoid.

PROPOSITION VIII.

338. Theorem. *The ratio of the areas of two similar polygons is equal to the ratio of the squares of two homologous sides.*



Let O and O′ represent two similar polygons.

To prove that $\frac{O}{O'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$, *or the square of the ratio of similitude.*

Suggestion 1. Divide the polygons into similar $\triangle$s, *1* similar to *1′*, etc. § 309.

2. What does the ratio $\frac{\triangle 1}{\triangle 1'}$ equal in terms of $\frac{AB}{A'B'}$? of $\frac{AC}{A'C'}$? $\frac{\triangle 2}{\triangle 2'}$ in terms of $\frac{AC}{A'C'}$? Compare $\frac{\triangle 1}{\triangle 1'}$ with $\frac{\triangle 2}{\triangle 2'}$. Compare $\frac{\triangle 2}{\triangle 2'}$ with $\frac{\triangle 3}{\triangle 3'}$, etc. § 337.

3. Compare $\frac{\triangle 1}{\triangle 1'}$, $\frac{\triangle 2}{\triangle 2'}$, $\frac{\triangle 3}{\triangle 3'}$, etc.

4. Compare $\frac{O}{O'}$ with any of the ratios in (3) § 284.

5. State the ratio $\frac{O}{O'}$ in terms of homologous sides (Sug. 2).

Therefore—

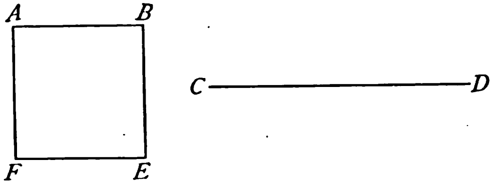
339. Corollary.—*The ratio of the areas of two similar polygons equals the ratio of the squares of any two homologous lines.*

Name all the cases in which the ratio of areas has been found.

Notice that ratio of areas is always expressed by the ratio of the products of two factors. If there is a common dimension this factor may be canceled out of both terms of the ratio. State all the cases in which this is true; in which the ratio remains a product. When may this product be stated as a square?

Proposition IX.

340. **Problem.** *Upon a given line as base, to construct a rectangle equal in area to a given square.*



Let C D be the given line, and A E the given square.

To construct upon C D as a base, a rectangle equal in area to the square A E.

Suggestion 1. Let X be tne required altitude. Hence $AB \times AB = CD \times X$.

2. Make a proportion from the above equation, having X for the fourth term. § 263.

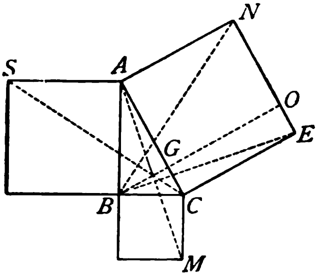
3. What problem is involved to find X? § 317.

4. Construct the rectangle.

Therefore —

Proposition X.

341. Theorem. *The square described upon the hypotenuse of a right triangle is equal to the sum of the squares described upon the other two sides.*



Let A B C represent a right triangle whose hypotenuse is A C, A E the square upon the hypotenuse, and B S and B M the squares on the other two sides.

To prove that A E is equal to the sum of B S and B M.

Suggestion 1. Draw $B G \parallel$ to $C E$, and extend it to meet $N E$ at O. Draw $B N$, $B E$, $C S$, and $A M$. What kind of polygons are $C O$ and $A O$? Why?

2. Compare $\triangle$s $B C E$ and $A C M$.

3. Compare the area of $\triangle$ $A C M$ with the area of the square $B M$, and the area of $\triangle$ $B C E$ with the area of the rectangle $C O$. § 332.

4. Compare the area of the square $B M$ with the area of the rectangle $C O$.

5. Compare $\triangle$s $C A S$ and $N A B$.

6. Compare the square $B S$ with the rectangle $A O$.

7. Compare the sum of the areas of the squares $B M$ and $B S$, with the sum of the areas of the rectangles $C O$ and $A O$; i. e., with the area of the square $C N$.

Therefore —

342. SCHOLIUM I. This proposition is known as the Pythagorean proposition. It is so named in honor of Pythagoras, who is supposed to have given the first demonstration of it.

343. SCHOLIUM II. The "square of a line" is an expression often used instead of the square described upon a line, and when so used means a plane surface in the form of a square whose side is the given line. But when this is used algebraically to express the area of the square surface whose side is the given line, it must be interpreted as in § 329.

(*a*) The Pythagorean proposition is often stated: The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides. It is frequently expressed as an algebraic equation, thus:

$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, in which the terms may be interpreted as pure numbers.

Ex. 215. If the diagonals of a quadrilateral intersect at right angles, show that the area of the quadrilateral is one-half the area of a rectangle whose sides are equal to the diagonals of the quadrilateral.

Ex. 216. To draw a common tangent to two given circles.

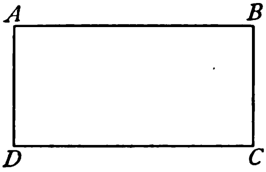
Ex. 217. If three equal circles are tangents to one another, the lines joining their centers form an equilateral triangle.

Ex. 218. Draw a circle with its center at a given point tangent to a given circle. When is there only one solution? When two?

Ex. 219. Construct a triangle with an area equal to 9 times the area of a triangle whose sides are 6, 7, and 9.

PROPOSITION XI.

344. Problem. *To construct a square equal in area to a given rectangle.*



Let A C be a given rectangle.

To construct a square equal in area to rectangle A C.

Suggestion 1. Let X represent the side of the square required.

2. Express the area of AC in terms of base and altitude; of the required square in terms of X.

3. Make an equation, and from it a proportion, and find X. §§ 263, 319.

NOTE—There are three previous propositions by which the value of X can be found. It may be time well used by pupils to work out more than one solution.

Ex. 220. Construct a square equal in area to a given trapezoid.

Ex. 221. Construct a square equal in area to a given triangle.

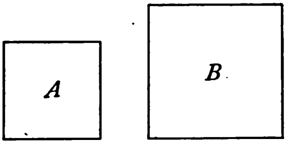
Ex. 222. Construct a square equal in area to a given parallelogram.

Ex. 223. If the hypotenuse of a right triangle is 15 ft. and the ratio of the legs is $\frac{3}{4}$, what is the area?

Ex. 224. Prove that the area of an equilateral triangle constructed on the hypotenuse of a given right triangle, is equal to the sum of the areas of the equilateral triangles constructed on the other two sides of the given right triangle. §§ 341 and 337.

PROPOSITION XII.

345. Problem. *To construct a square whose area is equal to the sum of the areas of two given squares.*



Let A and B be two given squares.

To construct a square whose area is equal to the sum of the areas of A and B.

Suggestion. What proposition is suggested by the theorem?

PROPOSITION XIII.

346. Problem. *To construct a square whose area is equal to the difference of the areas of two given squares.* § 343 (*a*).

Ex. 225. Construct a square whose area is equal to the sum of the areas of three given squares.

Ex. 226. Upon a given line as a base, construct a rectangle whose area is equal to the sum of the areas of a given square, a given trapezoid, and a given triangle.

Ex. 227. The area of a square inscribed in a circle is one-half the area of a square circumscribed about the same circle.

Ex. 228. Construct a parallelogram having a given base and a given angle, whose area is equal to a given rectangle.

Ex. 229. Find the dimensions of a rectangle whose perimeter is 96 in. and whose area is 40 sq. in.

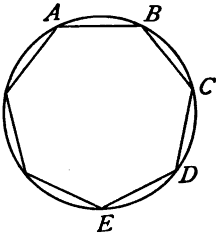
CHAPTER V.

REGULAR POLYGONS AND CIRCLES.

347. **A regular polygon** is a polygon which is both equilateral and equiangular. The equilateral triangle and square are regular polygons.

PROPOSITION I.

348. **Theorem.** *An equilateral polygon inscribed in a circle is a regular polygon.*



Let A D represent an equilateral polygon inscribed in a circle.

To prove that A D is a regular polygon.

Suggestion 1. What remains to be proved to make *A D* a regular polygon according to the definition, § 347?

2. Compare ∠s *A B C*, *B C D*, etc. Give auth.

3. Apply the definition of a regular polygon to *A D*.

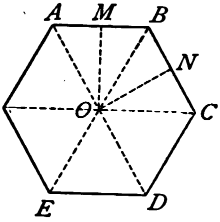
Therefore —

349. COROLLARY. *A regular polygon may have any number of sides.*

If the circumference of a circle be divided into any number of equal parts, the lines joining the points of division form an inscribed equilateral polygon. How does the number of sides of the polygon compare with the number of parts into which the circumference is divided?

Proposition II.

350. Theorem. *A circle can be circumscribed about a regular polygon. A circle can be inscribed in a regular polygon.*



Let A D represent a regular polygon.

I. *To prove that a circle can be circumscribed about A D.*

Suggestion 1. At M and N, the middle points of two adjacent sides, erect ⊥s and extend them until they meet, as at O. Why do they meet? O is the center of a circumference which passes through A, B, and C. Why?

2. To prove $O A = O D = O E$, etc.

(*a*) Join O with each vertex of the polygon. Compare $O C$ with $O B$ (Auth.); $\angle O B A$ with $\angle O C D$ (Auth.); $\triangle O B A$ with $\triangle O C D$, and hence, $O A$ with $O D$.

(*b*) The circumference through A, B, and C also passes through D. Why?

(*c*) In a similar manner show that the same circumference passes through E, etc. Can a ⊙ be circumscribed about the polygon $A D$?

II. *To prove that a circle can be inscribed in the polygon $A\,D$.*

Suggestion 1. Compare the distances of the various sides of the polygon from O. § 188.

2. Complete the demonstration.

Therefore —

QUERY: How many sides has polygon $A\,D$? See the theorem.

351. **The radius of a regular polygon** is the radius of the circumscribed circle.

352. **The apothem of a regular polygon** is the radius of the inscribed circle.

353. **The center of a regular polygon** is the center of the inscribed or circumscribed circle.

354. **The angle at the center of a regular polygon** is the angle formed by two radii drawn to two adjacent vertices of the polygon.

From the definitions just given the following corollaries may be deduced. Prove them.

355. COROLLARY I. *The angle at the center of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

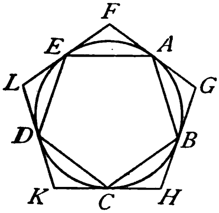
356. COROLLARY II. *An interior angle of a regular polygon is equal to the sum of all the interior angles of the polygon divided by the number of sides of the polygon.*

357. COROLLARY III. *The angle at the center of a regular polygon is the supplement of an interior angle of the polygon.*

358. COROLLARY IV. *The radius of a regular polygon bisects the angle of the polygon to which it is drawn.*

PROPOSITION III.

359. Theorem. *If a circumference is divided into any number of equal parts, the tangents drawn through the points of division form a regular circumscribed polygon.*



Let the circumference A B C, etc., be divided into equal parts, A B, B C, etc., and at the points A, B, etc., let tangents be drawn forming the polygon G H K, etc.

To prove that G H K, etc., is a regular polygon.

Suggestion 1. Compare ∠s G, H, K, etc. Ex. 130.

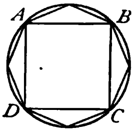
2. Compare lines $F A$, $A G$, $G B$, $B H$, etc. Exs. 64 and 125.

3. Compare lines $F G$, $G H$, $H K$, etc.

4. Apply definition of a regular polygon.

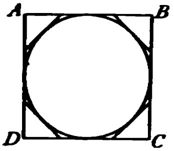
Therefore —

360. Corollary I. *If the vertices of a regular inscribed polygon are connected with the middle points of the arcs subtended by the sides, a regular inscribed polygon of double the number of sides is formed.* § 179.



361. Corollary II. *The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides.*

362. Corollary III. *If a regular polygon is circumscribed about a circle, and if tangents are drawn at the middle points of the intercepted arcs, a regular circumscribed polygon of double the number of sides is formed.* § 359.



363. Corollary IV. *The perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.*

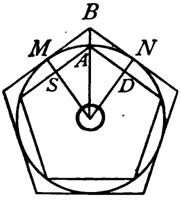
Proposition IV.

364. Theorem. *If a regular polygon is inscribed in a circle, and if tangents are drawn at the middle points of the arcs subtended by the sides of the inscribed polygon:*

I. *A regular circumscribed polygon is formed.*

II. *The sides of the regular circumscribed polygon are parallel to the sides of the inscribed polygon, each to each.*

III. *The vertices of the circumscribed polygon lie in the radii extended of the inscribed polygon.*



Let S A D represent a regular inscribed polygon; M, N, etc., the middle points of the arcs subtended by the sides; M B, B N, etc., tangents drawn through M, N, etc.

To prove —

I. *$M B N$ is a regular polygon.*

II. *$M B$ is parallel to $S A$, $B N$ is parallel to $A D$, etc.*

III. *B lies in $O A$ extended, etc.*

Suggestion for I. § 359.

Suggestion for II. Draw a radius $\perp$ to a side of the inscribed polygon and extend.

Suggestion for III. Draw $O M$ and $O N$ to adjacent points of tangency. Draw $O B$ and $O A$ to the vertices of the circumscribed and inscribed polygons, respectively, included between those points of tangency. Compare $\triangle$s $S A O$ and $D A O$; $\triangle$s $M B O$ and $N B O$.

2. $\therefore \angle M O N$ is bisected by $A O$ and also by $B O$. Why?

3. Complete the demonstration. § 47.

Therefore—

Ex. 230. The ratio of similitude of two similar polygons is $\frac{5}{7}$, and the sum of their areas is 518 sq. in. Find the area of each polygon. § 279.

Ex. 231. Find the base of a rectangle whose area is 108 sq. ft. and whose altitude is 6 ft. Compare your method with that of § 340.

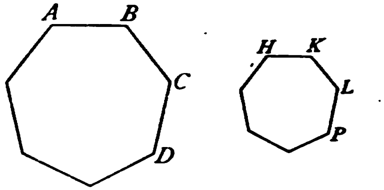
Ex. 232. Find the area of a right triangle whose hypotenuse is 1 ft. 8 in., and one of whose legs is 1 ft.

Ex. 233. The area of a polygon that circumscribes a circle equals one-half the product of its perimeter and the radius of the circle.

Ex. 234. If the middle points of two adjacent sides of a parallelogram be joined, a triangle is formed equal in area to one-eighth of the area of the parallelogram.

PROPOSITION V.

365. Theorem. *Regular polygons of the same number of sides are similar.*



Let A D and H P represent two regular polygons of the same number of sides.

To prove that A D and H P are similar polygons

Suggestion 1. What must be established to make the polygons similar?

2. Compare $\angle A$ with $\angle H$, $\angle B$ with $\angle K$, etc.

3. Compare $\frac{AB}{BC}$ with $\frac{HK}{KL}$, etc.; $\frac{AB}{HK}$ with $\frac{BC}{KL}$, etc.

Extend the series.

Therefore —

Ex. 235. What is the locus of centers of circles which are tangent to a given line at a given point?

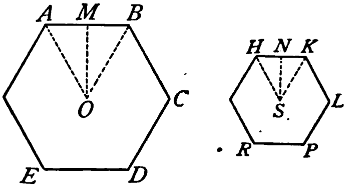
Ex. 236. Two parallel chords of a circle are, respectively, 36 inches and 48 inches long; the radius of the circle is 30 inches. What is the distance between the chords?

Ex. 237. Find the dimensions of a rectangle whose perimeter is 16 in. and whose area is 15 sq. in. Of one whose perimeter is 28 ft. and whose diagonal is 10 ft.

Ex. 238. If a parallelogram be inscribed in or circumscribed about, a circle, the diagonals pass through the center.

PROPOSITION VI.

366. Theorem. *The perimeters of two similar regular polygons have the same ratio as their radii, or their apothems.*



Let A D and H P represent two similar regular polygons.

To prove that $\frac{AB + BC + CD, \text{etc.}}{HK + KL + LP, \text{etc.}} = \frac{OA}{SH} = \frac{OM}{SN}$.

Suggestion 1. Compare the ratio of the perimeters with the ratio of any two homologous sides, as with $\frac{AB}{HK}$. Compare $\frac{AB}{HK}$ with $\frac{OA}{SH}$, with $\frac{OM}{SN}$. Give auth. § 305.

Complete the demonstration.

Therefore —

PROPOSITION VII.

367. Theorem. *The areas of two similar regular polygons have the same ratio as the squares of their radii, or the squares of their apothems.*

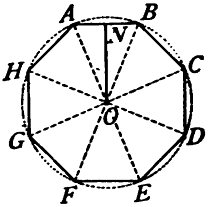
Let O and S represent two similar regular polygons.

To prove that $\frac{O}{S} = \frac{AO^2}{HS^2} = \frac{MO^2}{NS^2}$.

Suggestion. The same method may be used as in § 366. See § 338.

Proposition VIII.

368. Theorem. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Let A D be a regular polygon, O N its apothem, and A B C, etc., its perimeter.

To prove that the area of the polygon is equal to one-half the product of A B C, etc., and O N.

Suggestion 1. Circumscribe a circle, and draw radii to the vertices of the polygon.

2. Compare the distances from the center to the sides of the polygon. Give auth.

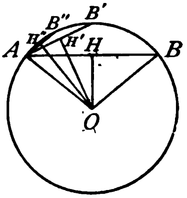
3. What is the area of △ *A O B*? Of △ *B O C*? etc.

4. What is the sum of the areas of all the △s?

Therefore —

Work out Sug. 4 algebraically as well as geometrically.

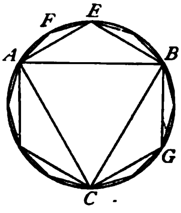
369. A **Lemma** is a proposition, corollary, or postulate inserted out of its natural order for the purpose of immediate use in demonstrating another proposition.



370. Lemma. *If the number of sides of a regular inscribed polygon be increased indefinitely the apothem is a variable which approaches the radius as a limit.*

OH, OH', OH'', etc., is a variable which approaches OA, or radius, as its limit.

371. Lemma. *If the number of sides of a regular inscribed polygon be increased indefinitely, the perimeter of the polygon is a variable which approaches the circumferance of the circle as a limit.*



Perimeter ABC, $AEBG$, etc., AFE, etc., is a variable approaching the circumference as its limit.

372. Lemma. *If the number of sides of a regular inscribed poloygon be increased indefinitely, the area of the polygon is a variable which approaches the area of the circle as a limit.*

Polygon ABC, $AEBG$, etc., AFE, etc., is a variable approaching the circle as its limit. § 211 (d) and (e).

Are §§ 371 and 372 true if the words circumscribed be substituted for inscribed?

Ex. 239. If one acute angle of a right triangle is 60°, prove that the area of the equilateral triangle constructed on the hypotenuse is equal to the area of a rectangle whose adjacent sides are the two legs of the right triangle.

Ex. 240. If two triangles have two sides of one equal respectively to two sides of another and the included angles supplementary, the triangles are equal in area.

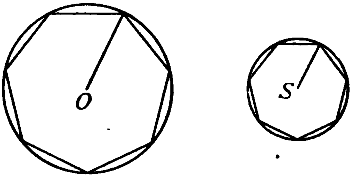
Ex. 241. Draw a circle through a given point tangent to a given line at a given point.

Suggestion. Find two loci of the center.

Ex. 242. The diagonals of a parallelogram divide it into four triangles equal in area.

PROPOSITION IX.

373. Theorem. *The circumferences of two circles have the same ratio as their radii.*



Let O and S represent two circles, R and R′ their radii, and C and C′ their circumferences.

To prove that $\frac{C}{C'} = \frac{R}{R'}$.

Suggestion. 1. Inscribe in the two circles similar regular polygons. Let P and P' represent their perimeters. Then $\frac{P}{P'} = \frac{R}{R'}$. Give auth.

2. From the equation in Sug. 1, $P = \frac{R}{R'} \times P'$. § 270, Note 1.

3. Now, let the number of sides of each polygon be indefinitely increased, always, however, keeping the number of sides of one polygon the same as the number of sides of the other polygon. During this change the variable P is always equal to the variable $\frac{R}{R'} \times P'$. Why?

4. Since P and P' are variables which approach C and C' respectively, as their limits (§ 371), therefore, P and $\frac{R}{R'} \times P'$, are variables which approach C and $\frac{R}{R'} \times C'$, respectively as their limits. Why? § 214 (*a*).

5. Compare C and $\frac{R}{R'} \times C'$. Give auth.

6. Compare $\frac{C}{C'}$ and $\frac{R}{R'}$.

Therefore —

QUERY: In Sug. 1, $\frac{P}{P'} = \frac{R}{R'}$. Is $\frac{R}{R'}$ a variable? Why? Is $\frac{P}{P'}$ a variable?

374. COROLLARY I. ***The circumferences of two circles have the same ratio as their diameters.***

If D and D' represent their diameters, $\frac{C}{C'} = \frac{D}{D'}$. Auth.

375. COROLLARY II. ***The ratio of the circumference of a circle to its diameter is constant.***

For, by Cor. I, $\frac{C}{C'} = \frac{D}{D'}$, and by alternation $\frac{C}{D} = \frac{C'}{D'}$. This ratio is represented by the Greek letter π (pi). Hence, $\frac{C}{D} = \pi$; or, $\frac{C}{2R} = \pi$.

376. COROLLARY III. $C = \pi D$; or, $C = 2\pi R$.

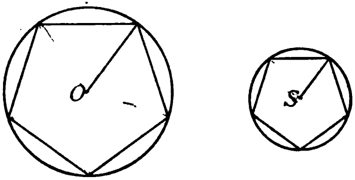
Ex. 243. What is the locus of the vertex of all isosceles triangles upon a given base?

Ex. 244. $A\,B\,C$ is a triangle, and D any point in $B\,C$ extended. Find a point E in $A\,B$ or $A\,B$ extended, such that the area of the triangle $E\,B\,D$ will be equal to the area of the triangle $A\,B\,C$. § 317.

Ex. 245. Draw a line through a given point in a side of a triangle so as to divide the triangle into two parts equal in area.

PROPOSITION X.

377. Theorem. *The areas of two circles have the same ratio as the squares of their radii.*



Let A and A′ represent the areas, and R and R′ the radii of the circles O and S, respectively.

To prove that $\frac{A}{A'}$ is equal to $\frac{R^2}{R'^2}$.

Suggestion 1. Inscribe in the two circles regular polygons of the same number of sides, and let M and M' represent their respective areas.

2. Proceed as in § 373.

Therefore —

378. COROLLARY I. *The areas of circles have the same ratio as the squares of their diameters.*

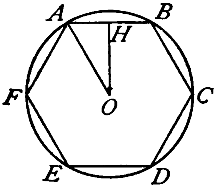
379. COROLLARY II. *The areas of similar sectors have the same ratio as the squares of their radii.*

Ex. 246. Two tangents to a circle whose radius is 8 inches are drawn from a point 12 inches from the center. Find the length of the chord joining the points of tangency.

Ex. 247. The legs of a trapezoid are each 15 inches, and the bases are 12 inches and 30 inches respectively. Find the area of the trapezoid.

PROPOSITION XI.

380. **Theorem.** *The area of a circle is equal to one-half the product of its circumference and radius.*



Let O represent a circle, R its radius, C its circumference and A its area.

To prove that A is equal to one-half the product of C and R. § 329.

Suggestion 1. Inscribe in the circle a regular polygon, and let P denote its perimeter, M its area and r its apothem.

2. What is the area of M in terms of P and r? § 368.

3. If the number of sides of the polygon be indefinitely increased, M is a variable. Why?

4. What is its limit? § 372.

5. To find the limit of $\frac{1}{2} P \times r$, the area of M:

(*a*) P is a variable. Why? What is its limit? Why?

(*b*) r is a variable. Why? What is its limit? Why?

(*c*) $P \times r$ is a variable whose limit is $C \times R$. § 214 (*c*).

(*d*) $\frac{1}{2} P \times r =$? Why? § 214 (*a*).

6. Compare the limits of the two equal variables, M and $\frac{1}{2} P \times r$.

$\therefore$ the area of $O =$? Why?

Therefore —

381. COROLLARY I. *The area of a circle is equal to π times the square of the radius.*

Substituting the value of C, (§ 376), $A = \frac{1}{2} C \times R, = \frac{1}{2} \times 2\pi R \times R = \pi R^2$.

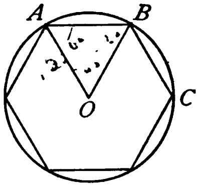
Commit to memory this formula for the area of a circle.

382. Corollary II. *The area of a sector is equal to one-half the product of its radius and arc.*

$\frac{\text{Sec.}}{\text{circle}} = \frac{\text{arc of sec.}}{\text{circumference.}}$ Hence, if a sector is $\frac{1}{n}$ of a circle, its arc is $\frac{1}{n}$ of its circumference; hence its area, $\frac{1}{n}$ of $\frac{1}{2} C \times R = \frac{1}{2}\,\text{arc} \times R$.

Proposition XII.

383. **Theorem.** *One side of a regular hexagon is equal to the radius of the circumscribed circle.*



∠ AOB = 1/6 of 4 rt ∠s
∠ BAO = ABO Because
then ∠s are [illegible]

Let A B C, etc., represent a regular hexagon inscribed in a circle whose radius is O A.

To prove $AB = OA$.

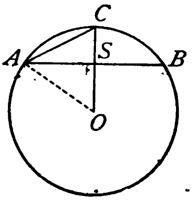
Suggestion 1. Draw radius OB and prove that ΔAOB is equiangular and equilateral.

Therefore —

Ex. 248. Describe a circle through a given point that shall touch a given circle at a given point. Find two loci of the center of the required circle.

PROPOSITION XIII.

384. Problem. *Given the radius of a circle and the side of a regular inscribed polygon, required to find the side of a regular inscribed polygon of double the number of sides.*



Let O represent a circle, R its radius, A B the side of a regular inscribed polygon and A C the side of a regular inscribed polygon of double the number of sides.

To find value of A C in terms of A B and R.

Suggestion 1. Connect O and A.

2. In $\triangle\, A\,S\,C$, express $A\,C$ in terms of $A\,S$ and $C\,S$ (§ 341); of $A\,B$ and $C\,S$.

3. Express $C\,S$ in terms of R and $S\,O$. Express $S\,O$ in terms of R and $A\,S$ (§ 341); in terms of R and $A\,B$.

4. Find value of $C\,S$ in terms of R and $A\,B$.

5. Find value of $A\,C$ in terms of R and $A\,B$. Reduce to simplest form, viz.: $A\,C = \sqrt{2R^2 - R\sqrt{4R^2 - \overline{A\,B}^2}}$

If R = unity, $A\,C = \sqrt{2 - \sqrt{4 - \overline{A\,B}^2}}$

Partial demonstration:

$$A\,C = \sqrt{\overline{A\,S}^2 + \overline{C\,S}^2} = \sqrt{\tfrac{1}{4}\overline{A\,B}^2 + \overline{C\,S}^2} =$$

$$\sqrt{\tfrac{1}{4}\overline{A\,B}^2 + (R - S\,O)^2} = \sqrt{\tfrac{1}{4}\,\overline{A\,B}^2 + \left(R - \sqrt{R^2 - \tfrac{1}{4}\overline{A\,B}^2}\right)^2} =$$

$$\sqrt{\tfrac{1}{4}\overline{A\,B}^2 + R^2 - 2R\sqrt{R^2 - \tfrac{1}{4}\,\overline{A\,B}^2} + R^2 - \tfrac{1}{4}\overline{A\,B}^2} =$$

$$\sqrt{2R^2 - 2R\sqrt{R^2 - \tfrac{1}{4}\,\overline{A\,B}^2}} = \sqrt{2R^2 - R\sqrt{4R^2 - \overline{A\,B}^2}}.$$

PROPOSITION XIV.

385. Problem. *To compute approximately the ratio of the circumference of a circle to its diameter.*

If $A\,C$ (§ 384) is one side of a regular polygon of n sides, $n \times A\,C$ = the perimeter which is an approximation of the circumference of the circumscribed circle, and the greater n is the closer the approximation. Why?

If $A\,C$ is one side of a regular hexagon, R being unity, the perimeter = 6. § 383.

If $A\,B$ (§ 384) = one side of a regular hexagon, $A\,C = \sqrt{2-\sqrt{4-(1)^2}} = \sqrt{2-\sqrt{3}} = .51763809$. Hence, the perimeter of a regular twelve-sided polygon = $12 \times .51763809 = 6.21165708$.

If the value of one side of the regular twelve-sided polygon be substituted for $A\,B$ in the formula, $A\,C$ is the value of one side of a regular twenty-four sided polygon.

By continuing the operation the perimeters of regular polygons of greater and greater number of sides may be found and consequently closer and closer approximations of the circumference in terms of radius.

In this way, the perimeter of a polygon of 768 sides has been computed to be 6.283169 +.

Dividing this result by the diameter, *i. e.*, by 2 (§ 375), gives 3.141584 +, as an approximate value of the ratio of the circumference of a circle to its diameter, an approximate value of π.

The approximate value usually used is 3.1416.

Therefore, $\pi = 3.1416$ approximately. *Q. E. D.*

(*a*) Some of the computed results are given in the following table:

No. Sides.	One Side.	Perimeter.
6	1	6.
12	$\sqrt{2-\sqrt{3}} = .51763809$	6.21165708
24	$\sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238$	6.26525722
48	$\sqrt{2-\sqrt{4-(.26105238)^2}} = .13080626.$	6.27870041
96	$\sqrt{2-\sqrt{4-(.13080626)^2}} = .06543817.$	6.28206396
192	$\sqrt{2-\sqrt{4-(.06543817)^2}} = .03272346$	6.28290510

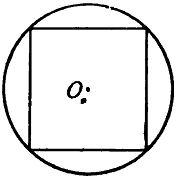
386. COROLLARY. *As an approximate value of π has been found, the area of a circle may be found approximately in terms of its radius. The approximate value is found by multiplying the square of the radius by 3.1416.* § 381.

387. SCHOLIUM. Archimedes (born 287 B. C.) found an approximate value of π. He proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$. The former of these two values is often used as an approximate value of π when great accuracy is not required.

In modern times the value of π has been computed to a large number of decimal places. Clausen and Dase, independently of each other, computed the value to the two-hundredth decimal place. Other computers have given the value to over five hundred decimal places, but their results have not been verified. The number is incommensurable, and cannot be expressed exactly by any number of decimal places.

PROPOSITION XV.

388. Problem. *To inscribe a square in a given circle*



Let O be a given circle.

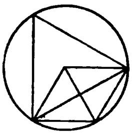
To inscribe a square in the circle O.

Suggestion. In any square, at what ∠ do the diagonals intersect?

389. COROLLARY. *By bisecting the arcs subtended by the sides of the square, and joining each point of division with the two adjacent vertices, a regular inscribed octagon is formed.*

Ex. 249. Measure the diameter and circumference of several articles in the form of a circle, as the end of a barrel, tin cup, etc., and divide the latter by the former to see how an approximation of π can be obtained.

Ex. 250. Compute the length of a perimeter of twenty-four sides and verify by the table, P. 211.



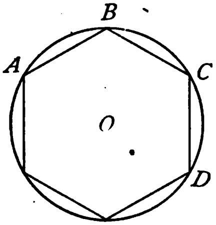
Ex. 251. Show that one side of an equilateral triangle inscribed in a circle is equal to $R\sqrt{3}$, or if R equals unity, $\sqrt{3}$.

Ex. 252. Use the formula, (§ 384), $\sqrt{2-\sqrt{4-ab^2}}$ in the case of a regular inscribed equilateral triangle to prove that the value of one side of a regular hexagon is **one.**

Suggestion. $AB = \sqrt{3}$. (Ex. 251.)

PROPOSITION XVI.

390. **Problem.** *To inscribe a regular hexagon in a given circle.*



Let O be a given circle.

To inscribe a regular hexagon in the circle O.

Suggestion. § 383.

391. COROLLARY I. *By joining the alternate vertices of the regular inscribed hexagon in order, an equilateral triangle is inscribed in the circle.*

392. COROLLARY II. *By bisecting the arcs subtended by the sides of a regular inscribed hexagon, and joining the points of division with the adjacent vertices of the hexagon, a regular dodecagon is inscribed in the circle.*

393. GENERAL SCHOLIUM. Methods have already been given of inscribing in a circle a regular polygon of 3, 4, 5, 6, 8, or 10 sides (see § 741). Any regular inscribed polygon being given, a regular inscribed polygon of double the number of sides can be formed by bisecting the arcs subtended by the sides and joining the points of division to the adjacent vertices of the given polygon. Hence, by means of the inscribed square regular polygons of 8, 16, 32, etc., sides can be inscribed; by means of the regular inscribed hexagon, regular polygons of 12, 24, 48, etc.,

sides can be inscribed; by means of the regular inscribed decagon regular polygons of 20, 40, 80, etc., sides can be inscribed. There is still one more set of polygons which can be inscribed in a circle. For, if from any point on the circumference of a circle a chord be drawn equal to one side of a regular inscribed hexagon, and from the same point another chord be drawn equal to the side of a regular inscribed decagon, the first chord subtends an arc which is one-sixth of the circumference, and the second chord an arc which is one-tenth of the circumference, and the difference between these two arcs is one-fifteenth of the circumference. Hence, the chord joining the extremities of the two chords previously drawn is one side of a regular inscribed polygon of fifteen sides, and from this figure regular polygons of 30, 60, etc., sides can be inscribed.

Until the beginning of the present century it was supposed that the polygons already enumerated were the only ones which could be inscribed by elementary geometry, but in a work published in 1801, Gauss proved, by means of the ruler and dividers only, that it is possible to inscribe regular polygons of 17 sides, of 257 sides, and in general, of any number of sides which can be expressed by $2^n + 1$, n being an integer, provided that $2^n + 1$ is a prime number.

Ex. 253. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

Ex. 254. Construct a triangle, given the base, an angle adjacent, and the distance from the vertex to the middle point of the base being given.

SUPPLEMENTARY PROPOSITIONS.

Ex. 433. (§ 96.)* The angle formed by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle at the base of the triangle.

Ex. 434. A line drawn from one end of the base of an isosceles triangle perpendicular to the opposite side makes with the base an angle equal to one-half the vertical angle.

Ex. 435. (§ 126.) ACB and ADB are two triangles on the same side of AB, such that AC is equal to BD, and AD is equal to BC, and AD and BC intersect at O. Prove that AOB is an isosceles triangle.

Ex. 436. If the vertical angle of an isosceles triangle is one-half as great as an angle at the base, the bisector of a base angle divides the given triangle into two isosceles triangles.

PROPOSITION I. (§ 214.)*

712. Theorem. *If a variable approaches zero as a limit, any product of the variable by a finite number approaches zero as its limit.*

Let V denote the variable which approaches zero as its limit, and let M denote any finite number.

To prove that MV *approaches zero as a limit.*

If MV cannot have zero as a limit, let S be any assigned quantity, however small, by which amount it differs from zero. As S is a finite magnitude, $\frac{S}{M}$ is a finite magnitude (Ax. 14, Cor.). Hence, V as it approaches O can be made smaller than $\frac{S}{M}$. (§ 211.) Hence, $M\ V$ can become smaller than S, but cannot equal zero. (Ax. 7, Cor.) Hence $M\ V$ can be made to differ from zero by less than any assigned quantity, as S. (Ax. 1.) Hence $M\ V$ approaches zero as a limit. § 212.

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* The references indicate where the Supplementary Propositions and Exercises may be inserted. They may be used anywhere after the reference but usually not before it.

Therefore—

PROPOSITION II.

713. Theorem. *If a variable approaches zero as a limit, any quotient of the variable by a finite constant approaches zero as a limit.*

Let V denote a variable that approaches zero as its limit, and let M denote any finite quantity.

To prove that $\frac{V}{M}$ *approaches zero as a limit.*

V has zero for its limit. Each value of $\frac{1}{M}$ of V is smaller than V (Ax. 14, Cor.), but cannot equal zero, hence must have zero for its limit. Ax. 9.

$\frac{V}{M} = \frac{1}{M}$ of V.

$\therefore \frac{V}{M}$ has zero for a limit.

Therefore—

PROPOSITION III.

714. Theorem. *The variable which is produced by multiplying a given variable by a finite constant has for its limit the product of the limit of the variable by the constant.*

Let V be a variable, L its limit, and M any finite quantity.

To prove $M\,V$ has $M\,L$ for a limit.

$L - V$ has zero for its limit. § 212.

$M\,(L-V)$, or $M\,L-M\,V$, has zero for its limit. § 712.

$M\,V$ has $M\,L$ for its limit. § 211.

Therefore—

PROPOSITION IV.

715. Theorem. *The variable which is produced by dividing a given variable by a finite constant has for its limit the quotient of the limit of the variable by the constant.*

Let V be any variable, L its limit, and M any finite constant.

To prove that $\frac{V}{M}$ has $\frac{L}{M}$ for its limit.

$L-V$ has zero for its limit. § 212.

$\frac{L-V}{M}$, or $\frac{L}{M}-\frac{V}{M}$, has zero for its limit. § 713.

$\therefore \frac{V}{M}$ has $\frac{L}{M}$ for its limit. § 211.

Ex. 437. In triangle $A\,B\,C$, angle B is three times A, and C is five times A; find each angle of the triangle.

PROPOSITION V.

716. Theorem. *If each of two variables has zero for its limit, their product has zero for its limit.*

Let V and V′ be two variables, each having zero for its limit.

To prove VV′ has zero for its limit.

M V has zero for its limit (§ 712) and cannot equal zero, and as V' can become less than M, VV' must approach zero as its limit. Why?

Therefore—

PROPOSITION VI.

717. Theorem. *If each of two or more variables has zero for its limit their sum has zero for its limit.*

Let V, V′, V″, etc., represent variables having zero for their limits.

To prove that $V + V' + V''$, etc., has zero for its limit.

Let S be any assigned quantity to represent the difference between V + V′ + V″, etc., and its respective limit.

V and V' can each become less than $\frac{S}{2}$ but cannot equal zero (Hyp.). Hence $V + V'$ can become less than S and cannot equal zero.

In the same way each of M variables can become less than $\frac{1}{M}$ of S. Hence, their sum can become less than S. As S is any assigned quantity, however small, $V + V' + V'' +$ etc., has zero for its limit.

Therefore—

PROPOSITION VII.

718. Theorem. *The sum of two or more variables, all increasing or all decreasing, has for its limit the sum of the limits of the variables.*

Let V, V′, V″, etc., represent variables and L, L′, L″, etc., their respective limits.

To prove $V+V'+V''+$ *etc., has for its limit* $L+L'+L''+$ *etc.*

$L-V, L'-V', L''-V''$, etc., each has zero for its limit. § 212.

Let R, R', R'', etc., represent $L-V$, $L'-V'$, $L''-V''$, etc., respectively.

Then $V=L-R$, $V'=L'-R'$, $V''=L''-R''$, etc. (§ 212), adding, $V+V'+V''+$ etc., $=L+L'+L''+$ etc., $-(R+R'+R''+$ etc.). Ax. 2. (*c.*)

$L+L'+L''+$ etc., $-V+V'+V''+$etc., $=R+R'+R''+$ etc. Ax. 3.

$R+R'+R''+$ etc., has zero for its limit. § 717.

$\therefore L+L'+L''+$ etc. $-(V+V'+V''+$ etc.), has zero for its limit.

$\therefore V+V'+V''+$ etc. has $L+L'+L''+$ etc., for its limit. § 211.

Therefore—

Ex. 438. If $A\,B$ and $A\,C$ are equal sides of a triangle, and if $B\,M$ and $C\,N$ are bisectors of angles B and C respectively, prove that triangles $A\,B\,M$ and $A\,C\,N$ are equal, and also that triangles $B\,C\,N$ and $C\,B\,M$ are equal.

PROPOSITION VIII.

719. Theorem. *The product of two variables both of which are either increasing or decreasing, has for its limit the product of the limits of the variables.*

Let V and V′ represent the two variables, L and L′ their respective limits.

To prove that $V V'$ has $L L'$ for its limit.

$L - V$ and $L - V'$ each has zero as its limit.

Let R and R' represent $L - V$, and $L - V'$ respectively.

$V = L - R$, $V' = L' - R'$.

$V V' = L L' - L' R - L R' + R R'$. Ax. 4.

$L L' - V V' = L' R + L R' - R R'$. Ax. 3.

$L' R$, $L R'$ and $R R'$ each has zero for its limit. Why?

$L' R + L R' + (-R R')$ has zero for its limit. Why?

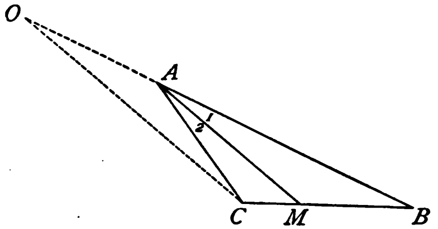
$L L' - V V'$ has zero for its limit. Why?

$V V'$ has $L L'$ for its limit. Why?

Therefore—

PROPOSITION IX. (§ 290.)

720. Theorem. *A line that bisects the vertical angle of a triangle divides the base into segments proportional to the two legs of the triangle.*



Let A C B represent a triangle, A M the line that bisects

the vertical angle of the triangle and C M and M B the segments of the base.

To prove $\frac{CM}{MB} = \frac{CA}{AB}$

Suggestion 1. Draw a line through C ∥ to $A\,M$ and extend $A\,B$ to meet it at O. Compare $A\,O$ and $A\,C$.

2. Compare $\frac{CM}{MB}$ with $\frac{OA}{AB}$; $\frac{CM}{MB}$ with $\frac{CA}{AB}$.

Therefore—

720 (a). When a point, as M, is taken in a line between its extremities, the line is **divided internally.** $A\,B = A\,M + M\,B$. When a point, as M', is taken in a line extended, the line is **divided externally** at the point. $A\,B = A\,M' + M'\,B$.

Verify the two equations. (Ax. 10.)

Suggestion. Let distance in the direction from A to B be positive, then distance in the direction from B to A is negative.

Ex. 439. Converse of Proposition IX. A line drawn through the vertex of a triangle that divides the opposite side into segments proportional to the other two sides, bisects the angle.

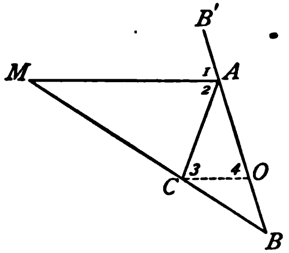
Use the figure in § 720 and prove the proposition.

Ex. 440. If $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A+B}{A-B} = \frac{C+D}{C-D}$.

Ex. 441. If the radius of one circle is the diameter of another, the circles are tangent to each other, and any line drawn from the point of contact to the outer circumference is bisected by the inner one.

PROPOSITION X.

721. Theorem. *If an exterior angle of a triangle is bisected, the side opposite is divided externally into segments proportional to the other two sides.*



Let A B C represent a triangle, B′ A C an exterior angle, and C M and M B the two segments into which the base is externally divided.

To prove that $\frac{CM}{MB} = \frac{CA}{AB}$.

Suggestion. Draw a line through C ∥ to the bisector, $A M$.

722. A line is divided in **mean and extreme ratio** if the ratio of the whole line to the greater segment is equal to the ratio of the greater segment to the lesser segment.

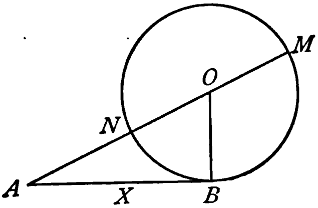
Ex. 442. State the converse of Proposition X.

Use figure § 721, and prove the proposition.

Ex. 443. If $A B C$ and $A B D$ are two triangles on the same base and on the same side of it, such that $A C$ equals $B D$, $A D$ equals $B C$ and $A D$ and $B C$ intersect at O, prove (1) that triangles $A B C$ and $A B D$ are equal in all respects; (2) that triangles $A O C$ and $B O D$ are equal in all respects; and (3) that triangle $A O B$ is isosceles.

PROPOSITION XI. (§ 314.)

723. Problem. *To divide a straight line in mean and extreme ratio.*



Let A B be a given straight line.

To divide A B in mean and extreme ratio.

Suggestion 1. At B erect a $\perp$ to AB equal to $\frac{1}{2}$ of AB, as BO.

With O as a center, and OB as a radius, describe a circumference. Draw a line through A and O, cutting the circumference at N and M.

On AB lay off a distance AX equal to AN.

2. $\frac{AM}{AB} = \frac{AB}{AN}$. Why? § 314. 4. $\frac{AB}{AN} = \frac{AN}{BX}$. Why?

3. $\frac{AN}{AB} = \frac{BX}{AN}$. Why? § 281. 5. $\frac{AB}{AX} = \frac{AX}{BX}$. Why?

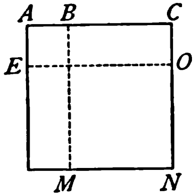
Therefore —

Ex. 444. A circle circumscribes an isosceles triangle, and tangents are drawn to the circle through the vertices of the triangle. Prove that these tangents form a second isosceles triangle, and that the two triangles cannot have equal vertical angles unless both are equilateral.

Ex. 445. In a given line determine a point which is equally distant from two given points not in the line.

PROPOSITION XII. (§ 330.)

724. Theorem. *The square described upon the sum of two lines is equal to the sum of the squares described upon the two lines plus twice the rectangle of the two lines.*



Let A B and B C be two given lines, and A C their sum. Let A N be the square described upon A C.

To prove that A N equals the sum of the squares described upon A B and B C, plus twice the rectangle whose sides are A B and B C.

Suggestion 1. Lay off $A E$ equal to $A B$, and draw $E O \parallel$ to $A C$; also draw $B M \parallel$ to $C N$.

2. Study the parts of square $A N$ and describe them in terms of the theorem.

725. SCHOLIUM. With the interpretation given in § 329 for the square of a line and the product of two lines, this proposition may be expressed thus:

If $A C = A B + B C$

then $\overline{A C}^2 = \overline{A B}^2 + \overline{B C}^2 + 2\, A B \times B C$.

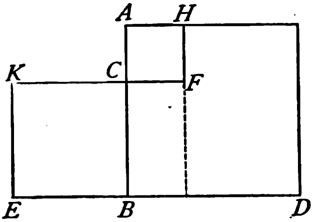
Compare this with the algebraic formula for the square of the sum of two numbers.

Therefore—

Ex. 446. Square 36, (30 + 6), by § 725 and compare your process, part by part, with the usual method.

PROPOSITION XIII. (§ 330.)

726. Theorem. *The square described upon the difference of two lines equals the sum of the squares described upon the two lines minus twice the rectangle of the two lines.*



Let A B and B C be two given lines, and A C their difference. Let A F be the square described upon A C.

To prove that A F equals the sum of the squares described upon A B and B C, minus twice the rectangle whose sides are A B and B C.

Suggestion 1. Let $A\ D$ be the square described upon $A\ B$, and $K\ B$ the square described on $B\ C$.

2. Compare square $A\ F$ + twice $(A\ B \times C\ B)$ with $\overline{A\ B}^2 + \overline{B\ C}^2$.

State conclusion in form of theorem.

Therefore —

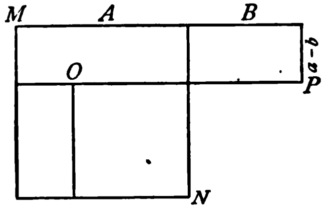
727. SCHOLIUM. With the interpretation given in § 329 for the square of a line and the product of two lines, this proposition may be expressed thus:

If $A\ C = A\ B - B\ C$, $\overline{A\ C}^2 = \overline{A\ B}^2 + \overline{B\ C}^2 - 2\ A\ B \times B\ C$.

Ex. 447. Through two points an inch apart draw two parallel lines.

PROPOSITION XIV. (§ 330.)

728. Theorem. *The product of the sum of two lines and their difference is equal to the difference of their squares.*



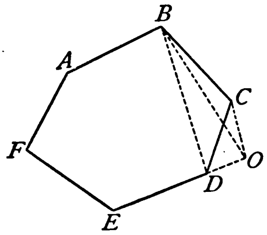
Let A and B represent the two lines, M N the square upon A, O N the square upon B and M P the rectangle (a + b) by (a – b).

To prove that M N minus O N is equal to M P.

Express algebraically. § 329.

PROPOSITION XV. (§ 336.)

729. Problem. *To construct a triangle equal in area to a given polygon.*



Let A B C, etc., be a given polygon.

To construct a triangle equal in area to the polygon A B C, etc.

Suggestion 1. Draw a diagonal, as *B D*, cutting off one vertex, as *C*.

2. Through *C*, the vertex cut off, draw a line ∥ to the diagonal *B D*.

3. Extend *E D* until it meets the line ∥ to the diagonal, as at *O*. Connect *O* with *B*.

4. Compare the areas of the △s *C B D* and *O B D*.

5. Compare the areas of the polygons *A B C*, etc., and *A B O E*, etc.

6. Compare the number of sides of the last polygon with that of the original polygon.

Continue the process until the polygon is reduced to a triangle.

Therefore—

Ex. 448. The altitude of a trapezoid is 3 ft. and the bases are 8 and 12 ft. respectively. Extend the non-parallel sides until they meet, and find the areas of the two triangles of which the trapezoid is the difference.

Ex. 449. (§ 341.) If the center of each of two equal circles lies on the circumference of the other, the square on the common chord is equal to three times the square on the radius.

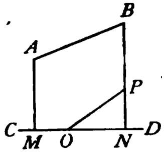
Ex. 450. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

Ex. 451. What is the ratio of the areas of two similar triangles whose homologous sides have the ratio $\frac{3}{5}$.

Ex. 452. Construct a pentagon. Find the number of square inches in its area by § 729 and § 336 and compare results.

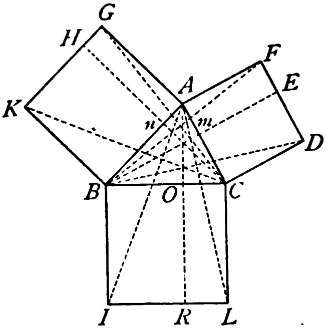
730 (a). The **projection of a point** upon a straight line is the foot of the perpendicular from the point to the line.

(b). The **projection of a straight line** upon another straight line is that part of the second line included between the projections of the extremities of the given line. The projection of A upon $C D$ is M, of B is N; of $A B$ upon $C D$ is $M N$, of $O P$ is $O N$.



PROPOSITION XVI. (§ 341.)

731. Theorem. *The square upon the side opposite an acute angle of a triangle equals the sum of the squares upon the other two sides minus twice the product of one of the two sides by the projection of the other side upon that side.*



Let A B C be a triangle of which the angle A is acute, and let B L, A D, and A K be the squares described upon the sides B C, A C, and A B, respectively.

To prove that the square B L equals the square A D *plus square A K minus twice A B times An the projection of A C on A B.*

Suggestion 1. From A, draw $A\,O \perp$ to $B\,C$, and extend it to $I\,L$. From B, draw $Bm \perp$ to $A\,C$, and extend it to $F\,D$. From C, draw $Cn \perp$ to $A\,B$ and extend it to $K\,G$.

2. Draw $A\,I$, $A\,L$, $B\,D$, $B\,F$, $C\,K$ and $C\,G$.

3. Compare rectangle $C\,E$ with rectangle $C\,R$, $H\,B$ with $B\,R$. See method § 341.

Square $B\,L$ with rectangles $H\,B + C\,E$.

4. Compare square $B\,L$ with squares $A\,K + A\,D$.

5. Compare rectangles $A\,H$ and $A\,E$.

6. $A\,H = A\,B \times$ the projection of $A\,C$ upon $A\,B$. Give auth.

Complete the demonstration.

Express the theorem algebraically.

Therefore —

Ex. 453. Construct a triangle equal in area to a given triangle, two sides of the required triangle being given. Show when there are two solutions, when one solution, and when no solution.

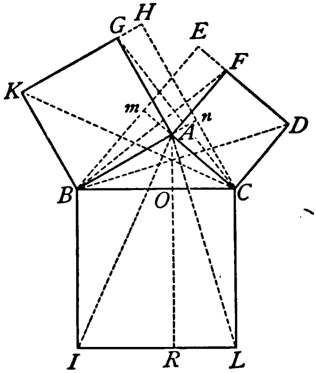
Ex. 454. If, from a point without a circle, two secants be drawn whose external segments are 8 inches and 7 inches, while the internal segment of the latter is 17 inches, what is the internal segment of the former?

Ex. 455. The sides of a triangle are 5, 6, and 7, and the side corresponding to 6, in a similar triangle, is 36; find the other two sides of the triangle.

Ex. 456. Two isosceles triangles have equal vertical angles. Prove that they are similar.

PROPOSITION XVII. (§ 341.)

732. Theorem. *The square upon the side opposite an obtuse angle of a triangle is equal to the sum of the squares upon the other two sides plus twice the product of one of the sides by the projection of the other side upon that side.*



Let A B C be a triangle, of which the angle A is obtuse; let B L, A K, and A D be squares upon the sides B C, A B, and A C, respectively, and let An be the projection of A C upon A B.

To prove that the square B L is equal to the sum of the squares A K and A D, plus twice the product of A B by An.

Suggestion 1. From each vertex of the △ draw ⊥s to the opposite side of. the opposite square, or the side extended, as *C H*, *A R*, and *B E*. Draw from the same points lines to the opposite vertices of the opposite squares, as *C K*, *C G*, *B F*, *B D*, *A I* and *A L*.

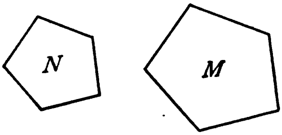
2. See method in § 731, and complete the demonstration.

Compare the demonstration of the propositions in sections 341, 731 and 732.

State one method of construction so that it may apply to each of the three propositions.

PROPOSITION XVIII. (§ 338.)

733. Problem. *To find two straight lines which have the same ratio as two given similar polygons.*



Let M and N be two given similar polygons.

To find two lines whose ratio is equal to the ratio $\frac{M}{N}$.

Suggestion 1. What is the ratio of M to N in terms of their sides? § 338.

2. Where has the ratio of two squares been compared to two lines? Ex. 209.

3. Make the required construction.

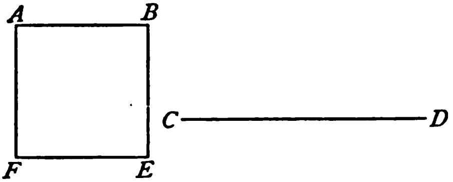
Ex. 457. If three similar polygons be constructed on the three sides of a right triangle, prove that the area of the polygon constructed on the hypotenuse equals the sum of the areas of the polygons constructed on the other two sides.

Suggestion. See §§ 341 and 338.

Ex. 458. All equal chords of any circle are tangents to some other circle.

PROPOSITION XIX. (§ 344.)

734. Problem. *To construct a rectangle in which the sum of the base and altitude is equal to a given line, and the area is equal to the area of a given square.*



Let C X equal the altitude, X D the base of the rectangle and A E the given square.

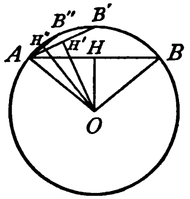
To find X in line C D.

Suggestion 1. $CX \times XD = AB^2$ or $AB \times BE$.

2. Make a proportion from above equation, and complete the solution. § 310, II.

PROPOSITION XX. (§ 369.)

735. Theorem. *If the number of sides of a regular inscribed polygon is increased indefinitely, the apothem is a variable which approaches the radius as a limit.*



Let A B be a side of a regular polygon, and O H its apothem.

To prove that if the number of sides of the polygon is increased indefinitely, OH is a variable which approaches OA as a limit.

Suggestion 1. What are the tests for the limit of a variable? § 211 (*a*).

2. In $\triangle AOH$ compare OA and OH; $OA - OH$ with AH; $OA - OH'$ with AH', etc.; AH with AB, AH' with AB', etc.

3. By continually increasing the number of sides of the polygon, one side, as AB, may be made less than any assigned line, however short. *Ax.* 14, Cor. I.

4. $OA - OH$ is a variable. What is its limit? Why? Suggestions 2 and 3. What relation does OA sustain to OH? (§ 212.) Apply the four tests for the limit of a variable. § 211 (*a*).

Therefore —

Ex. 459. Find the side of a square equal in area to a rectangle whose sides are 6 and 9.

Ex. 460. Given two similar triangles, construct a third triangle similar to the other two whose area shall be equal to the sum of the areas of the other two.

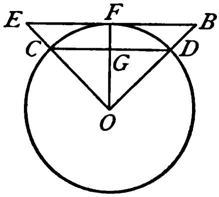
Ex. 461. Find the altitude of an equilateral triangle whose side is 10 inches. Find the side when the altitude is 10 inches.

Ex. 462. Construct a parallelogram having a given angle, whose base and altitude together are equal to a given line, and whose area is equal to the area of a given square.

Ex. 463. Find the locus of a point equally distant from the circumferences of two equal non-intersecting circles.

PROPOSITION XXI. (§ 370.)

736. Theorem. *If the number of sides of a regular inscribed polygon be increased indefinitely, the perimeter of the polygon is a variable which approaches the circumference of the circle as a limit.*



Let C D represent the side of a regular inscribed polygon and p its perimeter. Let C be the circumference of the circumscribed circle whose center is O.

To prove that as the number of the sides of the polygon increases, p is a variable which approaches the circumference of the circle as a limit.

Suggestion 1. Let EB be the side of a circumscribed polygon similar to the inscribed polygon, and let P denote the perimeter of the circumscribed polygon. Let r denote the radius and a the apothem of the inscribed polygon.

2. $\frac{P}{p} = \frac{r}{a}$. Give auth. *3.* $\frac{P-p}{P} = \frac{r-a}{r}$. Give auth.

4. $P-p = (r-a) \times \frac{P}{r}$. Give auth.

5. $\frac{P}{r}$ continually decreases, since P decreases and r remains unchanged; also $r-a$ diminishes indefinitely (§ 735). Hence $(r-a) \times \frac{P}{r}$ diminishes indefinitely, but cannot be made absolutely zero. (§ 712.) Hence $P-p$ diminishes.

6. $c-p$ is less than $P-p$, but still cannot be made equal to zero, therefore, c is the limit of p. § 212.

∴ The circumference of the ⊙ is the limit of the perimeter of the inscribed polygon when the number of sides of the polygon is made to increase indefinitely. *Q. E. D.*

Apply the four tests of the limit of a variable in drawing your conclusion.

737. Corollary. The circumference of a circle is the limit of the perimeter of the regular circumscribed polygon if the number of its sides be indefinitely increased. § 362.

Ex. 464. In $A\,B$, the diameter of a circle, or, in $A\,B$ extended, take any point C, and draw $C\,D$ perpendicular to $A\,B$; if A be joined with any point P, in $C\,D$, and $A\,P$ meet the circumference at Q; then $A\,P \times A\,Q = A\,C \times A\,B$. That is $A\,P \times A\,Q$ is a constant.

Show the varying positions of $A\,P$.

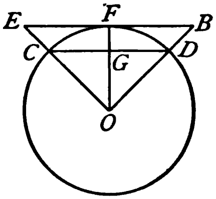
Ex. 465. If A is a given point, and P any point in a given straight line, and if a point Q be taken in the line joining A and P, so that $A\,P \times A\,Q$ is constant, then, as P moves along the given line, Q will move on the circumference of a circle which passes through A.

Ex. 466. If $A\,B$ be the diameter of a circle, and if a point P be taken on any chord $A\,Q$, or $A\,Q$ extended so that $A\,P \times A\,Q$ is constant, the locus of P is a straight line perpendicular to $A\,B$.

Ex. 466 (*a*). The sum of the squares of the diagonals of any quadrilateral equals twice the sum of the squares of its two diameters.

PROPOSITION XXII. (§ 371.)

738. Theorem. *If the number of sides of a regular inscribed polygon is increased indefinitely, the area of the polygon is a variable which approaches the area of the circle as a limit.*



Let C D represent a side, A the area of an inscribed polygon and M the area of the circle.

To prove that if the number of sides of the polygon is indefinitely increased, A is a variable whose limit is M.

Suggestion 1. Let r denote the radius and a the apothem of the inscribed polygon, also, let p denote the perimeter of the inscribed polygon, P the perimeter of a similar circumscribed polygon, and A^1 its area. Let EB be a side of the circumscribed polygon homologous to the side CD of the inscribed polygon.

2. Trapezoid $EBDC = (EB + CD) \times \frac{1}{2}(r - a)$. Why?

3. $A^1 - A = (P + p) \times \frac{1}{2}(r - a)$. Why?

4. $(P+p) \times \frac{1}{2}(r - a) < P \times (r - a)$. Why?

5. As the number of sides of the polygon is increased, P diminishes, and $(r - a)$ approaches zero as a limit (§ 735). Therefore, $P \times (r - a)$ approaches zero as a limit. Why? $\therefore A^1 - A$ continually decreases and approaches zero as a limit. Why?

But $M - A$ is always less than $A^1 - A$. As $M - A$ cannot become zero, M is the limit toward which A is approaching.

Apply the four tests of a limit of a variable in drawing the conclusion.

Therefore —

739. Corollary. *The area of the circle is the limit of the area of a regular circumscribed polygon if the number of sides of the polygon is indefinitely increased.*

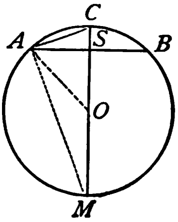
Ex. 467. Find the locus of a point equally distant from the circumferences of two concentric circles.

Ex. 468. Show how to cut off the corners of an equilateral triangle in such a way that the remaining figure will be a regular hexagon.

Ex. 469. (§ 384.) Another method for § 384.

To find value of $A\,C$ *in terms of* $A\,B$ *and* R.

Draw $C\,M$, a diameter. Connect A and M.



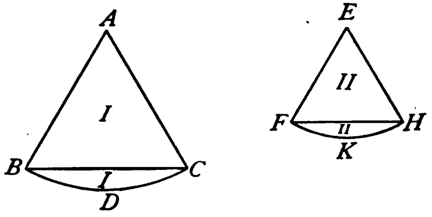
Suggestion. Find value of $A\,C$ in terms of $C\,S$ and $C\,M$. § 310.

Ex. 470. Find the area of a circular ring between the circumferences of two concentric circles whose diameters are eight and ten inches, respectively.

Ex. 471. Find the length of a line, Exercise 470, that is the tangent of one circle and the chord of the other.

PROPOSITION XXIII. (§ 379.)

740. Theorem. *The areas of two similar segments have the same ratio as the squares of their radii.*



Let B D C and F H K, Segment I and Segment II, respectively, represent two similar segments of circles whose centers are A and E.

To prove $\frac{\text{Seg. I.}}{\text{Seg. II.}} = \frac{\overline{AB}^2}{\overline{EF}^2}$.

Suggestion 1. Compare $\triangle$ I with $\triangle$ II.

2. $\frac{\text{Sec. I.}}{\text{Sec. II.}} = \frac{\triangle \text{ I.}}{\triangle \text{ II.}}$ § 379.

3. Take by alternation and that result by division.

4. $\frac{\text{Seg. I.}}{\text{Seg. II.}} = \frac{\overline{AB}^2}{\overline{EF}^2}$. Why?

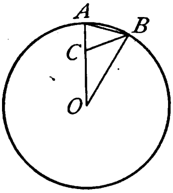
Therefore —

Ex. 472. If three equal circles are tangent to each other what is the area of the surface included between them if the radius of each of the circles is five rods?

Ex. 473. A hexagon each of whose sides, is six inches is inscribed in a circle. Find the area of a regular triangle inscribed in the circle.

Proposition XXIV. (§ 390.)

741. Problem. *To inscribe a regular decagon in a given circle.*



Let O be a given circle and O A its radius.

To inscribe a regular decagon in circle O.

Suggestion 1. Divide the radius OA in mean and extreme ratio, and let OC be the greater segment.

2. Draw the chord AB equal to OC and connect O with B, and C with B. To prove arc AB = one-tenth of the circumference.

3. $\frac{OA}{OC}=\frac{OC}{AC}$. Why? § 723. *4.* $\frac{OA}{AB}=\frac{AB}{AC}$. Why?

5. $\triangle OAB$ is similar to $\triangle CAB$. Why? § 303.

6. Compare $\angle ABC$ and $\angle O$; $\angle CBO$ with $\angle O$; $\angle ABO$ with $\angle O$.

7. $\angle O$ is contained how many times in 2 rt. $\angle$s? In 4 rt. $\angle$s?

8. Hence AB subtends what fractional part of the circumference?

9. State method of inscribing a regular decagon in a circle.

Corollary. *By joining the alternate vertices of a regular inscribed decagon, a regular inscribed pentagon is formed.*

2.5

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